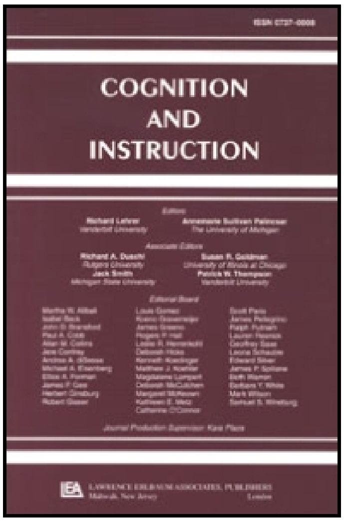

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Supporting Generative Thinking About the Integer Number Line in Elementary Mathematics

Geoffrey B. Saxe[a]; Darrell Earnest[a]; Yasmin Sitabkhan[a]; Lina C. Haldar[a]; Katherine E. Lewis[a]; Ying Zheng[a]
[a] University of California, Berkeley

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Supporting Generative Thinking About the Integer Number Line in Elementary Mathematics

Geoffrey B. Saxe, Darrell Earnest, Yasmin Sitabkhan, Lina C. Haldar, Katherine E. Lewis, and Ying Zheng
University of California, Berkeley

This report provides evidence of the influence of a tutorial "communication game" on fifth graders' generative understanding of the integer number line. Students matched for classroom and pretest score were randomly assigned to a tutorial ($n = 19$) and control group ($n = 19$). The tutorial group students played a 13-problem game in which student and tutor each were required to mark the same position on a number line but could not see one another's activities. To resolve discrepant solutions, tutor and student constructed agreements about number line principles and conventions to guide subsequent placements. Pre-/posttest contrasts showed that (a) tutorial students gained more than controls and (b) agreement use predicted gain. Analyses of micro-constructions during play revealed properties of student learning trajectories.

The number line is a geometric interpretation of number, a representation of a straight line measurable in linear unit intervals and constituted by a set of mathematical conventions. The conventions include the use of tick marks to partition intervals and indicate points, the placement of lesser to greater numbers from left to right, and the use of end arrows to indicate that numbers continue indefinitely. Though in the elementary grades the number line is a grounding representation in curriculum on integers and fractions, the measurement principles that underlie number line conventions are rarely the focus of mathematical inquiry (an exception is TERC's Investigations, Economopoulos, Wittenberg, Schifter, Russell, Murray, & Bastable, 2008). Many students leave elementary school with limited understandings of numerical and linear units (National Council for Education Statistics (NCES), n.d.; NCES, 2009), ill prepared for foundational topics in the secondary grades such as the interpretation of bar charts or graphing problems involving the coordinate plane (two perpendicular number lines). This article reports an investigation of a tutorial approach with fifth graders that provides one avenue to support students' generative understanding of number line principles for the representation of integers.

A premise for our study was that mathematical conventions emerge as people engage with coordination problems—problems in which concerted action is necessary. This idea is at the crux of treatments of convention from diverse perspectives, including epistemological (Lewis,

Correspondence should be addressed to Geoffrey B. Saxe, 4315 Tolman Hall, UC Berkeley, Berkeley, CA 94720-1670. E-mail: saxe@berkeley.edu

1969), linguistic (Croft, 2000; Keller, 1994), and psychological (Clark, 1996; Saxe & Esmonde, 2005a; Sfard, 2008). In our study, we extend the treatment of conventions to a communication game approach to tutoring number line principles and definitions. In our tutorial method, students and tutors solved problems that required them to mark positions on number lines without visual access to one another's activity, and then compared their solutions and their reasoning. To resolve discrepant solutions, tutor and student constructed agreements about number line principles and conventions that guided subsequent placements for new problems. Examples of these principles included order (e.g., numbers increase from left to right) and unit interval (e.g., distances between consecutive numbers must be the same). The sequence of tutorial problems was designed to engage the tutor and student in progressively more challenging ideas, requiring them to coordinate prior agreements, explore the entailments of these agreements, and establish new ones. The overarching goal of the study was to determine the effectiveness of the tutorial approach as well as the dynamic processes that mediate student learning.

STUDENTS' INTERPRETATIONS OF THE NUMBER LINE

Most students have been exposed to number lines in the course of instruction in elementary mathematics. But, as noted earlier, when assessed with numerical representations on the line, many students show limited understandings of mathematical conventions and principles. Some researchers have interpreted younger students' representations of whole numbers as logarithmic—as numbers increase they are closer together on the line (Booth & Siegler, 2008; Ramani & Siegler, 2008). Other studies document students' difficulties interpreting fractions on the number line (Saxe, Shaughnessy, Shannon, Langer-Osuna, Chinn, & Gearhart, 2007) or irregular sequences of whole numbers on the number line (Saxe, Shaughnessy, Gearhart, Haldar, Earnest, & Sitabkhan, 2009).

In one recent two-study investigation, Saxe et al. (2009) examined the character of students' developing conceptual coordinations of numerical units (whole numbers on the line) and linear units (concatenated congruent line segments constituting the line). The present study builds upon this prior work, and therefore we summarize those findings here. In the first Saxe et al. study, students were asked to order and position three integers on an unmarked (or "open") number line. In some tasks, the integers were consecutive (e.g., 0, 1, and 2; or 5, 6, and 7); in other tasks, the three integers were non-consecutive, and some of the numerical distances were irregular (e.g., 9, 10, and 13; or 9, 12, and 13). Most students placed the integers in appropriate ascending order from left to right, and most represented the numerical differences as linear unit distances for the consecutive integer tasks correctly (e.g., 0, 1, 2). But many treated the irregular sequences as if they were regular ones without coordinating numerical units (points represented by written numerals) and geometric units (congruent line segments). For example, one pattern in students' representations was equidistant placement of an irregular number sequence (e.g., 9, 10, 13), a pattern that respected the ordinal properties of the numbers but not a coordination of numerical units with linear units.

In the second Saxe et al. (2009) study, students were presented number lines with two points labeled with whole numbers and were asked to position a third whole number. The numbers for some tasks were consecutive, and the third number to be placed was the next consecutive number in a sequence (0 and 1 labeled, place 2). But the number sequence used in the other tasks were irregular (e.g., 9 and 11 labeled, place 12). As in the first study, students' solutions were more likely to be correct when students were locating successive numbers that differed by 1. But when placing numbers in an irregular sequence, many students' solutions were partial coordinations of numerical units, linear units, and order. For example, given values such as 9 and 11 and asked to place 12, students would often treat the 9→11 interval as a unit interval, placing the 12 where the 13 should be.

FRAMEWORKS USED TO GUIDE THE DEVELOPMENT OF OUR TUTORIAL INTERVENTION

A challenge for mathematics educators is to develop techniques to support students' use of their potentially *generative* but only partial knowledge to construct more integrated and coordinated understandings (diSessa, 1988; diSessa & Roschelle, 1994; Lehrer, 2003; Piaget, 1973). To address this challenge, we used two frameworks in the design of a tutorial approach. The first framework is both curricular and pedagogical: The goal is to articulate core number line conventions and principles as well as ways of supporting their construction in a tutorial. The second framework is cognitive: The focus is on the conceptual processes entailed in students' efforts to build on their partial but generative ideas (like numerical order, length, cardinal number) to construct the constitutive conventions of the line as a representation for numerical magnitudes. Of course, these frameworks are related, and we coordinate them in our tutorial design as well as in our analyses of student learning.

Curricular and Pedagogical Framework

In elementary mathematics instruction, linear models serve two general functions. One function is a *model for magnitudes external to the line*, such as a thermometer as a model for temperature, a bathroom scale as a model for weight, or a ruler as a model of linear distance. Such models and their constitutive conventions afford the recording of physical phenomena in linear form, calibrated in units that reflect physical magnitude. For example, in the case of scales for weight, the record is calibrated in units that do not descend below zero, and upper bounds are defined in relation to what is functional for the items measured (e.g., grams vs. kilograms). Once represented, models for physical magnitudes can be the object of arithmetical transformations and support meaning making, like additions of positive values of weight by moving a corresponding number of weight units to the right on a number line. Children's familiarity with physical magnitudes (length, weight, number, elevation, temperature) can be helpful representational contexts that support their intuitions about units and linear models (Ball, 1993; Lampert, 2001; Lehrer, 2003).

The second function of the number line is *self-referential*. The line can be used to reason about numerical relations without reference to any magnitude in the world beyond the line itself. For example, we can conceptualize the number 3 as defined by a point equidistant from 2 and 4, or 3 as the difference between 9 and 12, or 1 and 4, or 6 and 9, and so on. It is the self-referential use of the line that carries with it a number of necessary consequences (entailments), like negative number (the subtraction of a greater number from a lesser number), the symmetry of positive and negative numbers in relation to 0, and the infinite extension of numbers to the right and the left of the depicted number line.

Our premise is that engaging students in the construction of the constitutive conventions of the line by resolving communicative breaches is a useful pedagogical strategy in which an exploration of the modeling function of the line can support the elaboration of the self-referential function. In the initial phase of the tutorial, student and tutor solved modeling problems like marking the length of four red Cuisenaire Rods on an open number line with only zero represented and no other tick marks. A subsequent problem engaged student and tutor with modeling problems of multi-unit intervals when, for example, they marked the distance of 6 reds using purple rods (2 reds is the equivalent length of 1 purple). As students recorded rods of specified lengths, the rods served the functional equivalent of units, and thus the measurement and modeling activities supported students as they calibrated the line with tick marks.

As the tutorial progressed, the tasks engaged students with units defined on the line itself, and the line was treated as a self-referential numerical object constituted by the communicative agreements with no reference to rods at all. During the final phase of the tutorial, the tutor engaged the student in reflections on the mathematical entailments of the constitutive conventions of the number line, supporting the student's use of prior agreements about order and unit to construct further entailments like negative number and the symmetry of negative and positive integers around zero.

Cognitive Framework

As students constructed number line representations in our tutorial sessions, their prior understandings of number and the geometry of the line regulated their efforts. To ground our developmental analysis of students' construction of representations, we adapted a framework from prior research related to the microgenesis of representational activity (Saxe, 1991; Saxe & Esmonde, 2005a, 2005b; Saxe et al., 2009). Here, we apply that framework to the conceptual regulations involved in the modeling functions and the self-referential functions of the number line.

Modeling Function. We argue that three interwoven strands of conceptual activity regulate efforts to model a physical magnitude on a number line over microgenetic time (Saxe, 2004). To illustrate these, we consider a problem in which student and tutor are required to represent a distance of four Cuisenaire Rods on a number line with only 0 marked, as depicted in the *problem conditions* row in Figure 1. Accomplishing this task requires (1) quantifying the rods, (2) quantifying the line, and (3) coordinating the two processes of quantification to produce a specific numerical representation.

1. Quantifying rods. The rods presented in the problem conditions, like any physical objects, are not inherently magnitudes or units of length. They must be quantified in conceptualizing activity, each rod treated as a segment of length that can be joined to another. In Figure 1, the length is four red rods. Such quantifications were investigated by Piaget and colleagues in their treatment of geometry (Piaget, Inhelder, & Szeminska, 1960), and their findings revealed conceptual coordinations involved in the geometric translations of lengths and the additive compositions of lengths.

2. Quantifying the line. Like the rods, the line presented in the problem conditions is not inherently a numerical representation. There is no inherent directionality to a line (magnitudes increase from the left to the right), and tick marks on the line have no intrinsic numerical meaning.

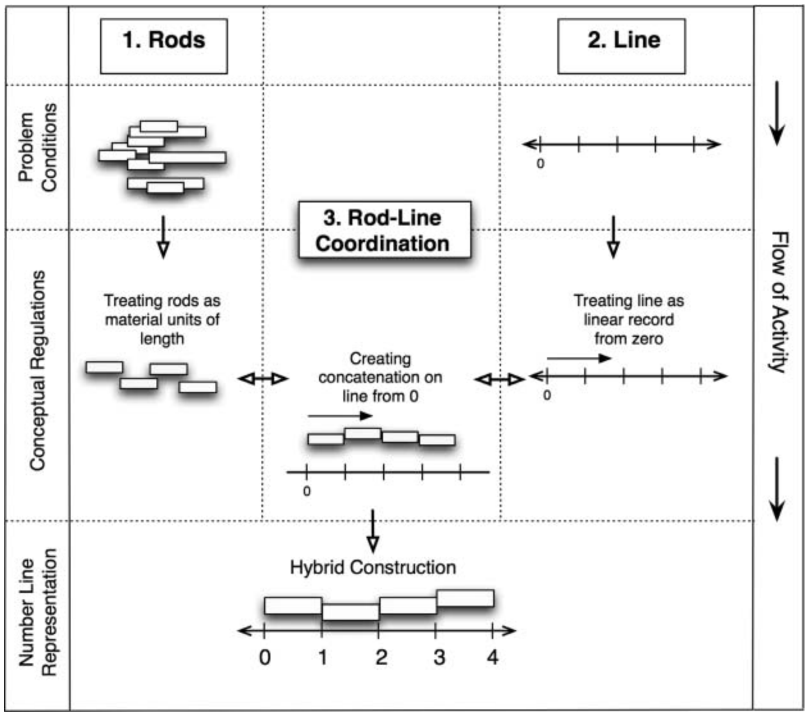


FIGURE 1 Three strands of conceptual activity regulating the construction of the line as a record of four red rods.

A line becomes a means of recording a linear representation of rods—like a distance of four red rods—when it is treated conceptually, as divisible into linear units that begin at an origin (zero).

3. Rod-line coordination. Even if (a) rods are treated as segments that together constitute a length and (b) the line is conceptualized as having an origin and as being divisible, these two conditions in themselves are insufficient to generate a number line representation. These two strands of conceptual activity must be coordinated, informing one another as the activity takes shape. In this generative process, conceptualizing the rods, the line, and their coordination must bootstrap one another (represented by the double arrows across the three activity strands). The rods, for example, must be conceptualized as a concatenated or iterated sequence of four segments that can be put in alignment with a line. Similarly, the line must be treated as a vehicle to accumulate linear concatenated segments or iterations that begin at the origin 0 and proceed to the right.

Self-Referential Function. Three strands of conceptualizing activity also constitute the self-referential use of the line. In contrast to modeling, however, linear units do not originate via correspondences with rods external to the line. Instead the strands of conceptualization shift to internal relations on the line. Consider Figure 2 that shows, under the initial problem conditions,

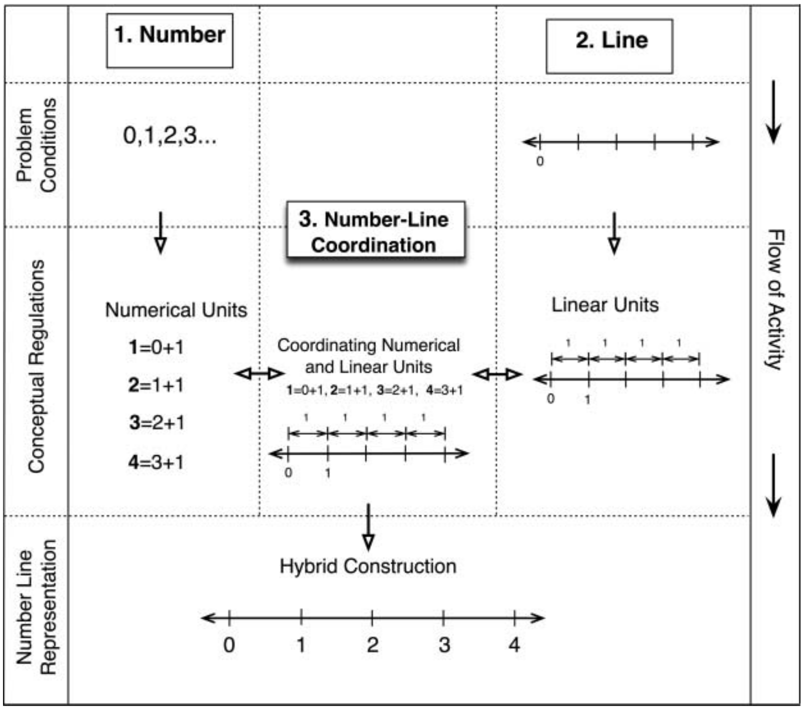


FIGURE 2 Three strands of conceptual activity regulating the use of the line as a self-referential representation for the task, "Mark where 4 goes on the number line."

a line with only 0 and 1 marked, labeled with tick marks; the task is to mark the number 4 on the line. The three interwoven strands of conceptual activity are: (1) a treatment of numerical units as additive such that numbers can be composed and decomposed, (2) a treatment of the line as susceptible to subdivision or concatenation using congruent segments, and (3) the coordination of numerical and linear units in the creation of a length of four units that draws upon the first and second strands as resources (double arrows across the three activity strands).

TUTORIAL DESIGN

To support an understanding of the number line as constituted by mathematical principles and conventions, we drew on the two frameworks to devise the problem-based communication game. In the game, all problems required student and tutor to coordinate their activity such that they placed points on identical number lines, but the activity of each was occluded from the other (although unbeknownst to the student, the tutor had a view of the student's activity—see Figure 3). To repair and avoid making discrepant placements, the tutor supported the construction of agreements (see Figure 4) like order (numbers become greater left to right) and unit interval (the distance between consecutive counting numbers must be the same). Over the course of the tutorial, student and

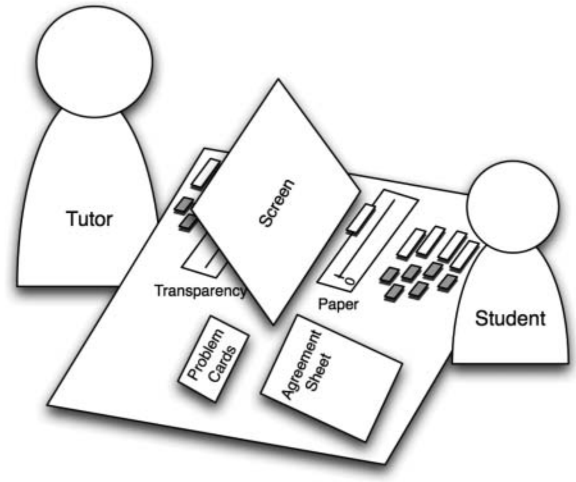


FIGURE 3 The communication game.

tutor solved 13 problem sets. Earlier problems involved exploring a linear magnitude—rods of different lengths coded by color (Cuisenaire Rods). Later problems involved the use of the line as a self-referential object. For each problem set, we developed three parallel forms so that, if a student and tutor had discrepant placements on one iteration, the tutor used the next parallel form. Our working hypothesis was that a student's mindful use of the agreements would lead to greater success at coordinated action; further, that the use of such mindful coordinations throughout our problem sequence would support a generative understanding of the entailments of the number line, like negative number, absolute value, and the line's symmetry.

Students participated in two sessions (on different days), with a set of wrap-up problems terminating each session. In each wrap-up, four non-routine number lines (e.g., with tick marks unevenly spaced, with numbers decreasing from left to right) were presented, and students were asked to evaluate whether each number line was "correct." In these wrap-up sessions, the tutor noted whether the students made spontaneous use of the agreements in appropriate ways to evaluate the adequacy of each line. If the student did not make use of agreements in appropriate ways, the tutor prompted the student with an appropriate use of the agreements. An expectation was that students' conscious and spontaneous use of appropriate agreements to mediate their evaluation of lines would be an important factor in supporting their learning during the tutorial.

By the end of the two tutorial sessions, students and tutors participated in all problem sets, though the number of iterations of each type varied across students. Over these two sessions, students and tutors generated five agreements (see Figure 4) and these were recorded on an Agreement Sheet. The first four agreements—Order, Unit, Multi-unit, and Every Number has a Place—were constructed during problems that involved the modeling function of the line with the rods. These agreements were then drawn upon again with the line alone as the tutorial proceeded, and the fifth agreement—Symmetry & Absolute Value—was constructed when negative numbers were generated on the line in the final phase of the tutorial.

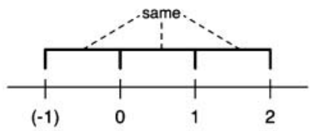
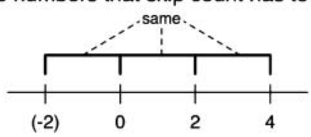
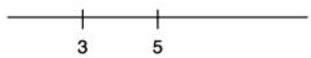
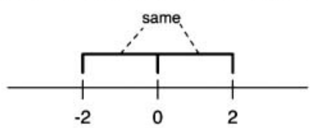
Principles (Agreements)	Written Agreement (to be Inscribed on Agreement Sheet)
a. Order	**Greater →** Numbers become greater from left to right. **← Less** Numbers become less from right to left.
b. Unit	**Count numbers - same distance.** The distance between consecutive counting numbers has to be the same 
c. Multi-unit	**Skip count numbers - same distance.** The distance between the numbers that skip count has to be the same 
d. Every number has a place	Every number has a place (and only one place) on the line but not all need to be shown 
e. Symmetry & Absolute value	**Positive & negative distances.** For every positive number there is a negative number that is the same distance from 0. 

FIGURE 4 Agreements generated between players to improve coordination of point placements.

The game had some affinities to previous methods referred to as teaching situations (Davydov & Tsvetkovich, 1991) and teaching experiments (Steffe, 2001; Tzur, 1999) as well as Vygotsky's method of double stimulation (Vygotsky, 1986). But unlike teaching situations and teaching experiments in which the focus is on students' construction of knowledge through tutorial situations and prompts, the purpose of the communication game was to follow the constructive process as student and tutor negotiated successful communications.

STUDY DESIGN

To determine the efficacy of the communication game on students' use and understanding of number line principles, we conducted an experimental study, randomly assigning matched fifth graders to tutorial and control conditions. We administered assessments related to number lines and integers to both groups. Our study addressed the following questions: (1) Was the communication game effective in supporting learning as determined by pre- to post-assessment gain of players as contrasted with controls? (2) Did variability in mindful agreement use during the tutorial predict learning gains? (3) What was the character of players' constructions and understandings as students moved from the recording/modeling function of the line to the self-referential function of the line?

METHODS

Participants

Ninety-five students in 6 fifth grade classrooms were administered a number line mathematics assessment consisting of 17 items (described below). Student performance on the assessment was used as a criterion to select a lower performing sample of students for participation in the study. The mean score for the total population of 95 students was 10.7 items correct ($SD = 3.3$). Forty students with scores at the lower half of the distribution (4 through 11 items correct) were matched in pairs based on number correct and then randomly assigned to a tutorial group ($n = 19$) and control group ($n = 19$).

Materials

Number Line Assessment. A 17-item assessment (Appendix) was used to evaluate students' knowledge of number line related properties and ideas: order, positive and negative integers, linear unit, absolute value and zero, and symmetry. Most of the items were based on interview tasks used in prior work to investigate student thinking about integers on the number line (Saxe et al., 2009).

Tutorial Materials. The following materials were used. (1) Two colors of pairings of Cuisenaire Rods, each with a 2:1 length relation: Red (length = 2 cm) and purple (length = 4 cm); light green (length = 3 cm) and dark green (length = 6 cm); (2) 13 sets of problem cards with 3 parallel forms in each set (39 problem cards in all); (3) a pre-printed number line for each of the 39 problems—the students' number lines were printed on paper, and the tutor's number lines were printed on transparencies; (4) wrap-up problems (four for each session) printed on sheets, four tasks per sheet. Additional materials included (5) a blank piece of paper (on which the tutor wrote number line agreements created in discussion with the student), (6) two markers (one for tutor and one for student), and (7) a removable screen.

Procedures

Pre- and Post-Assessments. The number line pre-assessment was administered to each participating classroom. The post-assessment was administered within several days following students' participation in the tutorial (tutorial group) or a similar duration of time (for the control group).

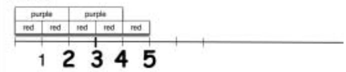
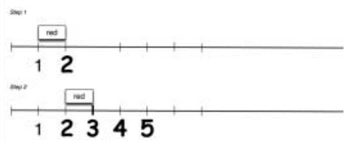
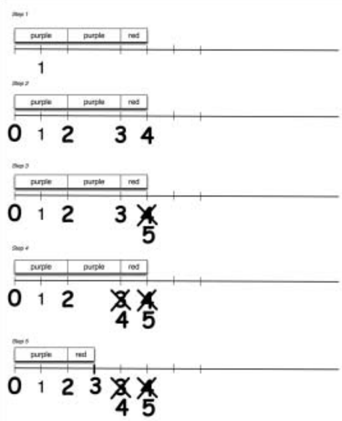
	Type of Coordination	Example for Problem V.1: "Mark where 5 reds is. You can use the red and purple rods."	Proportion of Students	
			Problem Set V (n=16)	Problem Set VI (n=15)
a.	Complete rod length used to unitize the line	Rod length constructed of either only unit rods or both multi-unit and unit rods. 	.38	.80
b.	Truncated rod length used to unitize the line		.38	.20
c.	Other: Multi-step/emergent construction		.25	.00

FIGURE 5 Multi-phase procedure used with each communication game problem.

Tutorial Overview. Each of the two tutorial sessions lasted about 45 minutes and was videotaped. For each session, a standard procedure was used as tutor and student engaged with successive problems. The multi-phased procedure is depicted in Figure 5a–d.

In the *problem card phase* (Figure 5a) the tutor presented the student with a card from the card deck, a number line keyed to the problem, and rods if required by the problem. In the *screen down phase* (Figure 5b), the occluding screen was positioned at a slight oblique angle so that the tutor could surreptitiously peer over the screen to see the child's workspace. In the *number line construction phase,* the student and tutor constructed number lines to address the problem card (Figure 5c). In the *screen up, positions match? phase*, the screen was removed once the constructions were completed (Figure 5d), and the tutor asked the student to overlay the two number lines, compare point placements, and explain his/her solution. During play, a video camera was positioned to record the child's activity and interactions with the tutor.

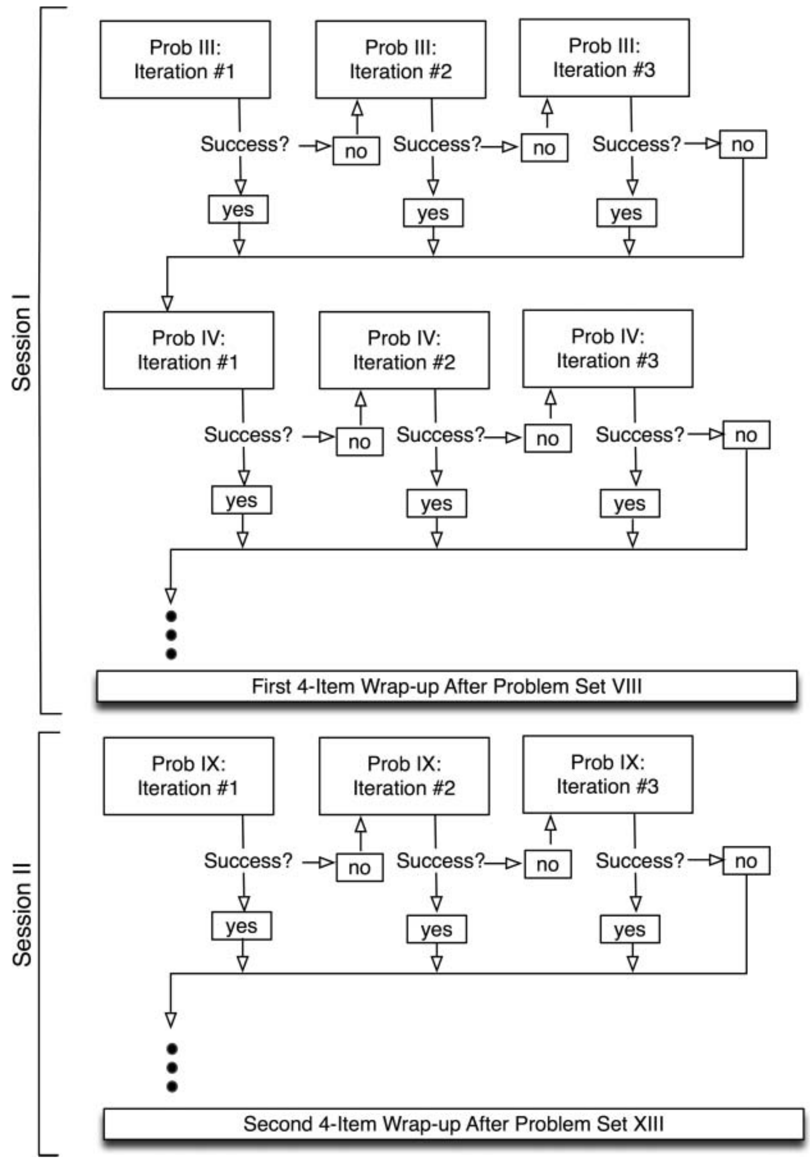


FIGURE 6 Sequencing of problem sets, iterations, and wrap-up items in the two tutorial sessions.

As shown in Figure 6, students had three opportunities (three iterations of a problem type) to achieve coordinated point placement. The tutor proceeded differently as a function of whether or not the tutor and students successfully matched point placements.

When the points did not match, the tutor supported joint reflection on sources of the discrepancy and ways of coordinating action on subsequent problems by following several heuristics, as appropriate:

1. *Refer back to agreements.* The tutor asked the student if he/she considered any of the agreements when placing the point and, if so, how. Then, drawing on the prior agreements to justify the placement, the tutor explained the placement of his/her own point (e.g., "I did mine this way because our agreements say ... ").
2. *Refer back to problem.* Sometimes a student's inappropriate construction could be interpreted as responding to a problem that was different from the one printed on the card. In such cases, the tutor oriented the child to the problem card stating, "I did mine this way because the card said to find ... ."
3. *Anticipate the next problem.* After drawing on the agreements or referring to the problem card, the tutor asked the student what each could do differently on the next problem card to try to place points at the same location.

If players reached the third iteration of a problem type and continued to have discrepant point placements, the tutor moved on to the next problem set. The student was unaware of the distinction between types of problems.

Figure 6 also depicts the procedure when the point placements matched. When the point placements matched, the tutor presented the student with a card from the problem deck that was the next problem type in the tutorial sequence. Thus the tutor navigated through the different problem versions, drawing cards of the same problem type or the next problem type depending on the dyad's (student's) success in achieving coordinate placements.

Tutorial Problem Sets. The 13 problem sets used over the two tutorial sessions are depicted in Figure 7. In our design of Session 1 problems (Problem Sets I through VIII), we were guided by three principles. The first was that earlier problems in the sequence (e.g., Problem Sets III and IV) should treat the line as a recording device in which units were not defined as a distance on the line but by the rods themselves (an "open line" with only one point labeled—as represented in Figure 1). In the later problem sets, the line should be a self-referential object in which the unit is defined by two labeled points (as represented in Figure 2 and exemplified in Problem Sets VII and VIII). The middle problems should serve a transitional function (Problem Sets V and VI).

The second principle was that problems should motivate agreements between tutor and student. The introductory problem sets were designed to provoke the Order and the Unit Distance agreements; Problem Sets III/IV were designed to provoke the Skip Counting agreement; Problem Sets V/VI were designed to provoke the Every Number has a Place agreement; and Problem Sets VII/VIII were designed to support the prior agreements to the treatment of the line as an autonomous, self-referential object.

The third principle that organized our design of Problem Sets III through VIII was that each problem set should be represented twice, with a less complex form immediately preceding a more complex form. For example, Problem Sets III and IV, in which the line was used to record Cuisenaire Rods, involved a translation from a problem in which students needed to represent unit Cuisenaire Rods in terms of multi-unit rods on the line (Problem Set III), to Problem Set IV, which involved a similar translation, but with the added complexity of coordinating unit and multi-unit rods. (See Figure 7 for more detail.)

In Session 2, the five problem sets included negative numbers, absolute value and symmetry (Problem Sets IX–XIII). For all problems in these problem sets, at least two points were labeled, and thus the line afforded the use of a self-referential representation.

Problem Focus & Number		Problem Statement and Corresponding Number Line	Rods (Physical Magnitude)	Line
Intro to rod use	II.3	Mark where 4 is. 0	Units (student's choice of rod)	No units defined on the line 0
Unit to multi-units quantified in rods	III	Mark where 6 reds is using the purple. 0	Unit (red) represented as multi-units (purple) rods purple / red red	No units defined on the line 0
	IV	Mark where 3 light greens is. Use BOTH the light green and dark green rods. 0	Unit (light green) represented as both the unit (light green) and multi-unit (dark green) rods dark green / light green light green	No units defined on the line 0
Unit to multi-units quantified in rods and on line	V	Mark where 5 reds is. You can use the red and purple rods. 1	Unit (red) represented as units (red) and/or multi-units (purple) rods purple / red red	Unit and multi-unit lengths defined on the line red purple 1
	VI	Mark where 4 light greens is. You can use the light green and dark green rods. 2	Unit (light green) represented as unit (light green) and/or multi-unit (dark green) rods dark green / light green light green	Unit and multi-unit lengths defined on the line Lt green Dark green Lt green 2
Unit to multi-units quantified on line	VII	Mark where 5 is. 0 1	Rods not used	Unit and multi-unit lengths defined on the line unit multiunit 0 1
	VII	Mark where 8 is. 5 7	Rods not used	Multi-unit lengths defined on the line multiunit 5 7
Constructing negatives with unit and multi-unit rods	IX	Mark where 2 reds less than 1 is on the number line using the rods 1 2	Unit (red) red	Units defined on the line red 1 2
	X	Using the purple rods, mark where 4 reds less than 2 is on the number line. 1 2	Unit (red) represented as multi-unit (purple) rods purple / red red	Units defined on the line red 1 2
Using symmetry, absolute value to construct positives and negatives	XI	Mark where 4 is on the number line without using the rods. -2 0	Rods not used	Multi-unit length defined on the line multiunit -2 0
	XII	Mark where -150 is on the number line without using the rods. 0 150	Rods not used	Multi-unit length defined on the line multiunit 0 150
	XIII	Mark where -1001 is on the number line without using the rods. 0 1000	Rods not used	Multi-unit length defined on the line multiunit 0 1000

FIGURE 7 Problem focus, problem number, and problem conditions (problem statement, rods used, and line used).

In our design of both Sessions 1 and 2, we ended with a wrap up phase. The tutor engaged the student with four wrap-up problems in each session to support agreement usage. In addition, at the beginning of the second session, the tutor and student reviewed the written agreements from the first session.

We present brief summaries of the 13 problem sets below—more complete descriptions are contained in Figure 7.

a. Problem Sets I and II—Introductory problems: The Order Agreement and the Same Distance Agreement. For the first two problem sets, the tutor purposely constructed a solution different from the student's. Uncoordinated solutions motivated the need for the first two agreements: Order (Problem Set I) and Counting Numbers–Same Distance (Problem Set II). For example, in Problem I.1, student and tutor were required to mark a point on their number lines, and the tutor ordered numbers on the line in the opposite direction from the student's order. The resulting discrepancy led to the need to negotiate an order agreement, typically an agreement that built on the student's understanding of the left to right order for increasing value of numbers. In problem II.2, the tutor intentionally used different colored rods than the student as a context to motivate the use of a common unit and the "Counting Numbers–Same Distance agreement." (Note that only on Problem Sets I and II did the tutor purposely place points in incorrect locations.)

b. Problem Sets III and IV: The Skip Counting Agreement and problems involving unit and multi-unit relations. The purpose of Problem Sets III and IV was to coordinate multi-unit to unit relations with rod lengths and record these coordinations on an unpartitioned line. For example, Problem III.1 required use of the 2:1 rod relations between purple and red; the directions stated, "Mark where 6 reds is using the purple rods," and the number line provided was marked only with zero. Problem IV.1 also required coordinated use of unit and multi-unit rods, though the rods shifted in lengths to light greens and dark greens.[1]

c. Problem Sets V–VIII: The Every Number Has a Place Agreement and moving from rod lengths on the line to internal relations between points on the line. The purpose of these problems was to support a transition from the line used as a means of recording the length of rods (Problem Sets III and IV) to the line as a self-referential object. Thus Problem Sets V and VI required the construction of rod lengths on the line, where the line was partitioned into unit and multi-unit intervals with a single non-zero point labeled. Problem Sets VII and VIII contained lines with two points labeled and thus did not make use of rods.

d. Problem Sets IX–XIII, The Absolute Value Agreement and negative numbers. The purpose of Problem Sets IX to XIII was to introduce negative numbers and explore ideas of absolute value and symmetry of numbers to the right and left of zero on the number line. Successful execution of these problem required players to extend existing agreements to numbers to the left of zero.

Wrap-Ups. Each of the two sets of wrap-up problems was linked to the mathematical content of the respective session. The wrap-up problems consisted of non-routine number lines printed on paper with boxes for the student to indicate whether the number line was "correct" or "incorrect"

[1]For problems that required use of Cuisenaire Rods (Problem Sets II–VI), we altered the colors (and hence the lengths) of the rods across problem sets. For Problem II the rods were of only one color, and the particular color choice emerged out of a cooperative agreement by the student and tutor. Problem Sets III and V required a red-purple pairing, and Problem Sets IV and VI required a light green-dark green pairing. For each rod pair, there was a 2:1 ratio in lengths between rod color (i.e., 2 reds = 1 purple; 2 light greens = 1 dark green). In Problem Sets VII and VIII, no rods were used.

Wrap-Up Problem		Agreement Targeted
Wrap-Up Session #1		
I.1	1 2 3	Unit and Multi-Unit
I.2	2 4 5	Unit and Multi-Unit; Every number has a place
I.3	1 2 4	Unit and Multi-Unit; Every number has a place
I.4	4 5 7	Unit and Multi-Unit; Every number has a place
Wrap-Up Session #2		
II.1	-1 -2 -3 -4	Order
II.2	-1000 0 1000	Unit and Multi-Unit; Symmetry and Absolute Value
II.3	-2 -1 1 2	Unit and Multi-Unit
II.4	-4 -2 -1	Unit and Multi-Unit; Every number has a place; Order

FIGURE 8 Agreements targeted for each wrap-up problem.

and to use the agreements to justify their answer orally. If students answered accurately and used a relevant agreement to justify the answer, the tutor moved on to the next wrap-up problem. If the student responded correctly and did not provide the relevant agreement to support it, the tutor asked explicitly about the relevant agreement. The complete set of number lines used in the wrap-up problems is contained in Figure 8 along with the targeted agreements.

RESULTS

The results are presented in four sections each linked to an organizing research question as depicted in Table 1. Across sections, our focus is on a convergent analysis of whether and how students' participation in the tutorial supported learning gains, although we draw on different data sources and use different kinds of analytic techniques in each section. First, we analyze the efficacy of the tutorial, contrasting learning gains of the tutorial and control groups. Second, we analyze whether tutorial students' use of agreements predicted learning gains. Third, to probe the character of learning in the tutorial, we analyze tutorial students' shifting conceptual coordinations as they worked through a sample of tutorial problems. Finally, we conclude with an analysis of a single student's progress through Session 1 problems to provide further insight into the dynamics of learning in the tutorial. We analyze his interaction with the tutor as agreements were made, discrepant placements were considered, and coordinated placements were accomplished.

TABLE 1
Overview of research questions, participants, data sources and analyses

Research Question	*Participants*	*Data sources*	*Analyses*
Was the tutorial effective?	Tutorial and Control students	Pre- and post-assessments	ANOVA contrasting tutorial and control groups gains Analysis of effect size from pre- to posttests
Did active and appropriate agreement use mediate learning gains?	Tutorial students	Pre- and post-assessments Video of students' solutions to and tutor student interactions on uptake problems	Appropriate and inappropriate agreement use on uptake problems Correlational analyses of appropriate/ inappropriate agreement use and learning gains
What were students' learning trajectories through problem iterations? —patterns of difficulty? patterns of successful constructions?	Tutorial students	Video of students solutions to Session 1 problems	Rod and number line coordinations that students produced in their successful and unsuccessful solutions to Session 1 problems
How did students make use of learning on prior problems to support learning on subsequent ones?	Tutorial student	Video of a student's and tutor's trajectory through Session 1 problems Students pre and post-assessment	Longitudinal analysis of a student's tutorial interactions, agreement use, and learning progress through Session 1 problems

Effects of the Tutorial: Pre- to Posttest Gains for Tutorial and Control Groups

To determine whether the tutorial game supported students' understanding of number line principles, we contrasted pre- and posttest scores of tutorial and non-tutorial groups. The group contrast served two purposes. First, the contrast provided a control for regression to the mean, a potential threat to validity; recall that students in the groups were the lower achieving students in their fifth-grade classrooms, and thus gains from pre- to posttest for the tutorial group could be an artifact of sampling, not an unbiased estimate of learning. Second, the random assignment of students to the tutorial and no-tutorial groups served to control for possible practice effects and classroom experiences that may have positively (or negatively) influenced game players' performance.

Figure 9 contains box plots that contrast the pretest and posttest performances of the tutorial game and no game groups. At pretest, groups showed a similar distribution of scores as expected, since pairs were matched on pretest and then randomly assigned to tutorial and control groups within classrooms. Mean scores at pretest were 7.95 ($SD = 1.54$) and 8.36 ($SD = 1.95$) for the tutorial and control groups, respectively. The amount of gain differed for the groups. For the tutorial group, the gain was 4.1 points ($SD = 2.26$) whereas for the control the mean gain was 1.4 points ($SD = 3.32$). For the tutorial group, the gain represented a shift of 1.82 standard deviations. A 2 $\times$ 2 ANOVA revealed an interaction between group and pre- to post-assessments (F (1,36) $= 8.83$ ($p < .005$), and follow-up analyses revealed a significant difference between pre- and posttest performance ($t(18) = 7.92$, $p < .0001$), but no difference between pre- and posttest for the control. Thus the tutorial group showed a significant gain, the gain was not due to a regression to the mean, and the effect size for the gain was large.

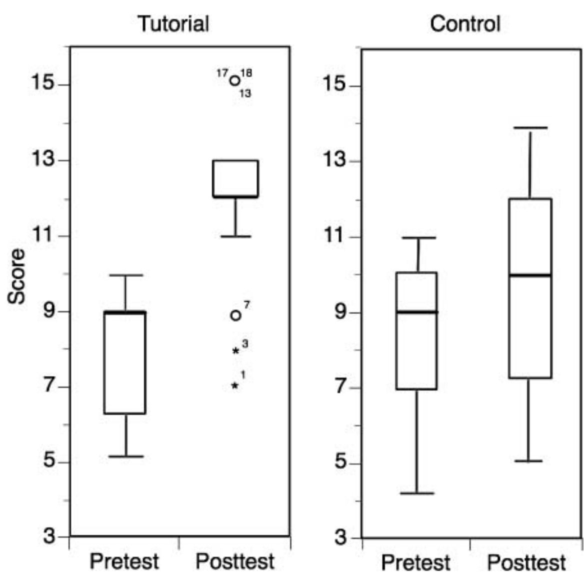


FIGURE 9 Box plots of pre- and posttest scores for tutorial and control groups.

A few students showed little or no improvement from pretest to posttest. Figure 9 depicts these students as outliers on posttest in the tutorial group: Student #1 and Student #3 received posttest scores of 7 points and 8 points respectively. Inspection of the tutorial records of these students suggests an explanation for their little or no improvement. In Session 1 of the tutorial, Student #1 achieved a pass score for each of the problems, effectively eliminating the student from the process of working through agreements with the tutor. Student #3, in contrast, did not achieve pass scores on most of the problem iterations and hence had a great deal of exposure to the game, but very limited appropriate use of the agreements. The student who was the less extreme bottom outlier in Figure 9 (Student #7) scored 9 on the posttest, gaining 3 points from pre- to posttest performance.

Mindful Use of Agreements and Learning Gains

The purpose of this analysis is to help explain the processes and practices that supported learning as revealed in the prior section. We expected that over the course of the two tutorial sessions, students would vary in whether they incorporated mindful use of the agreements in reasoning about number lines. Further, we expected that developing active use of agreements would support (and hence predict) learning gains. To produce an index of agreement use, we focused on the wrap-up sessions. For each wrap-up, students evaluated the adequacy of four lines keyed to the respective session's problems. For each line, we coded and analyzed (1) the correctness of students' judgment, (2) the extent of spontaneous appropriate and inappropriate agreement use, and (3) the relation between appropriate and inappropriate agreement usage and learning gains.[2]

Students' Judgments About Adequacy of Number Lines. Figure 10 and Figure 11 show the distribution of students' initial correct judgments by wrap-up problem for Sessions 1 and 2, respectively.[3] Student performances on the wrap-up problems varied by problem. Students had more difficulty with (a) problems involving the use of negative number and (b) problems that involved the coordination of order and unit distance agreements.

Spontaneous appropriate and inappropriate agreement use. After the student provided an initial judgment about whether a wrap-up number line was correct, the tutor asked for a justification. We coded the agreements that students referenced in their justifications and whether students used these agreements appropriately or inappropriately. *Appropriate agreement use* was coded when students spontaneously applied an agreement correctly, drawing on an agreement to argue appropriately that an adequate number line was correct or that an inadequate number line was incorrect. To illustrate, on Session 1, Problem 2, students were presented with a number line that included three numbers, 2, 4, and 5, with the distance between 2 and 4 twice that of the distance between 4 and 5. One student who received the *appropriate agreement use* code stated

[2]After the student's spontaneous evaluation of each line, if agreements were not used at all or used inappropriately, the tutor engaged the student with an evaluation of the line in relation to the previously established agreements. The procedure ensured that the evaluation was achieved using agreements appropriately by the conclusion of the session.

[3]One student did not respond on problem 1.

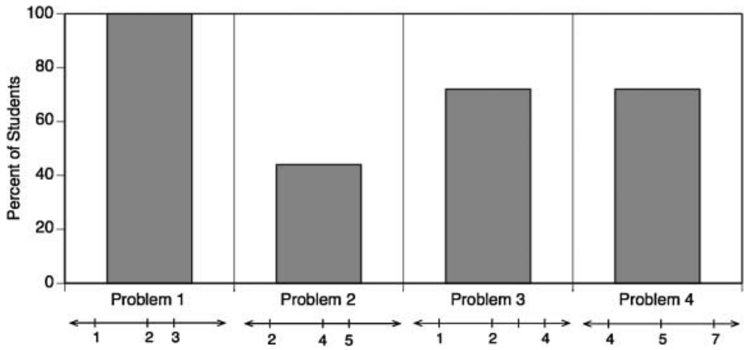


FIGURE 10 Percent of students producing correct initial evaluations of number lines by problem in Session 1.

that the number line was correct and justified her answer by referring to the agreement that states that every number has a place, but does not necessarily need to be shown. The student placed her finger where 3 would be and indicated that if 3 were present, the numbers would be equally spaced. To clarify further, she wrote in the 3 halfway between the 2 and 4 and again said "they're the same distance," referring to the unit distance agreement (counting numbers-same distance). In this case, the student successfully coordinated two agreements appropriately. *Inappropriate agreement use* was coded when students applied an agreement incorrectly to the number line, drawing on an agreement inappropriately to support a judgment that an adequate number line was incorrect or that an inadequate number line was correct. To illustrate, again consider Session 1, Problem 2. One student stated that the number line was incorrect. He justified his answer with an inappropriate use of the unit distance agreement (counting numbers-same distance), arguing that the 2 and 4 are further apart than the 4 and 5.

Two coders rated inappropriate and appropriate agreement usage during wrap-up sessions for four students initially. Inter-rater reliability was 94%. Coders then proceeded to code independently. To evaluate coder drift, midway through the coding process an additional student's wrap-up responses were coded by both raters, and reliability was 100%.

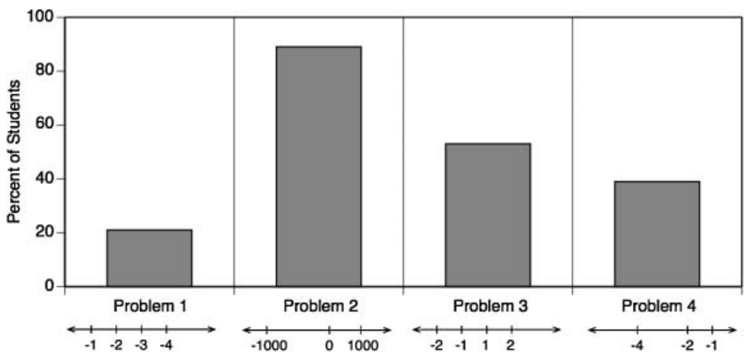


FIGURE 11 Percent of students producing correct initial evaluations of number lines by problem in Session 2.

Relation between agreement use and learning gains. The purpose of our analysis was to determine whether mindful use of agreements predicted learning gains. We used the frequencies of appropriate and inappropriate student agreement use over the course of the wrap-ups as estimates of the character of agreement use over the tutorial sessions. For the measure of appropriate agreements, we used total appropriate agreements on the more difficult target problems, since it was these problems that produced considerable variation in appropriate agreement use. These problems spanned varied kinds of principle violations, including order for negative number (Session 2, Problem 1) and equal distances for numerical unit (Session 2, Problem 3), or the problem design suggested a principle violation where there was none (Session 1, Problem 2; Session 2, Problem 4). For inappropriate agreement use, we used all of the problems in the computation of the measure, because inappropriate agreement use on these easier problems provided information on students' mindful use of agreements and contributed to variation in students' inappropriate agreement scores. To determine whether agreement use predicted learning gains, we correlated the two measures of agreement use (appropriate and inappropriate) with gains, controlling for pretest performance. The analyses revealed moderately strong correlations. Students who used agreements more appropriately in the wrap-up sessions were more likely to have higher gain scores (r ($N = 19$) = .56, $p = .008$), and students who used agreements more inappropriately were likely to have lower gain scores (r ($N = 19$) = –.67, $p = .001$).

Learning Trajectories Through the Tutorial

To analyze the process of student learning in the tutorial, we focused on Session 1 as these problems set the foundational agreements used throughout the tutorial. The purpose was to document the character of the coordinations students produced on each problem type, from the use of the line as a recording device as they translated unit rods into multi-unit lengths on open number lines with only one point labeled (Problem Sets III and IV), to open number line problems with the line constrained by additional tick marks (Problem Sets V and VI), to problems with two points labeled that focused on the line as an autonomous representation (Problem Sets VII and VIII).

Problem Sets III and IV: Unit to Multi-Unit Relations Quantified in Rods. For the iterations used in Problem Sets III and IV, the student and tutor were required to produce a unit-to-multi-unit translation of the rods on the number line. In Problem III.1, for example, tutor and student were required to represent a distance of 6 reds with purples on a line with only 0 marked. Recall that if the tutor judged the student's performance to be not passing, the tutor drew the next problem card that was the same problem type (e.g., III.1→III.2); if judged as passing, the tutor drew the card for the next problem type (e.g., III.1→IV.1). As always, the student was unaware that a judgment was being made or that the next problem card was contingent on a judgment.

The performances judged to be "passing" required the coordination of three strands of conceptual activity sketched in the introductory section (see Figure 1)—(a) quantification of the rods, (b) quantification of the line, and (c) their coordination in the construction of the endpoint of a rod-length on the line with 0. The quantification of the rods involved a translation of unit lengths into multi-unit lengths, as when a problem card required a length of 6 reds using only the purples (6 red rods is equivalent to 3 purple rods). The quantification of the line required treating the line as having a start point from the zero tick mark and a translation of a unit distance (or multi-unit distance) proceeding from left to right. The coordination of these two strands required the construction of a specified length, like a length of 6 reds with 3 purples starting from zero and proceeding to the right on the line.

To document the way students successfully coordinated their quantification of the rods and the line in relation to one another, we coded four types of passing coordinations that varied in sophistication. Figure 12 illustrates these four variations for Problem Set III; it also shows the proportion of students who used these coordination types for their final (passing) iteration for Problem Sets III and IV, respectively. The *n*s reflect the number of students who passed at any of the three iterations of each problem type.

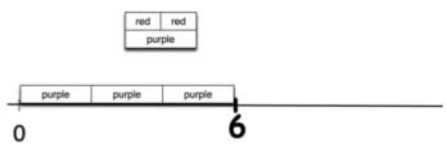
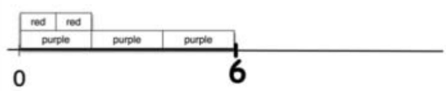
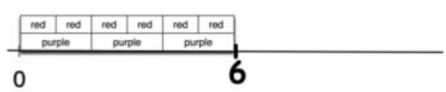
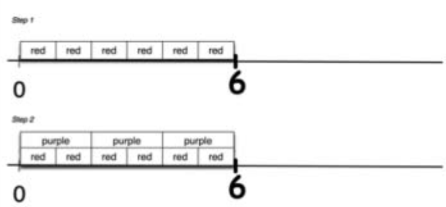
	Type of Coordination	Example for Problem III.1: "Mark 6 reds using the purples."	Proportion of Students	
			Problem Set III (n=17)	Problem Set IV (n=18)
a.	Many-to-one correspondence produced off the line, then used to construct unit lengths on the line.		.41	.39
b.	Many-to-one correspondence: Uses multi-unit and limited unit.		.18	.06
c.	Many-to-one correspondence: Uses multi-unit and unit exhaustively.		.23	.22
d.	Multi-unit used as an afterthought (not instrumental in the creation of the number line).		.18	.33

FIGURE 12 Proportion of students using each of four coordination types for their final (passing) iteration of Problem Sets III and IV.

Students' passing approaches reveal the way they generated hybrid representations of the rods and the line together to create a linear distance in rod units in order to identify a point. We observed ways in which the constraints of the line and the rods became interwoven in the process of students' constructions and in their representations. When students used a strategy that anticipated the need to coordinate units and multi-units with the rods to produce a representation on the line (see Figure 12a), students first established a 2:1 correspondence between reds and purples off the line with the rods alone, perhaps anticipating the linear relation "required" by the line. They then used the correspondence to construct the length of 6 reds using three purples, aligning the end of the first with 0 (extended to the right), and marking the number 6 at the end of the third purple. Other coordinations involved establishing the 2:1 correspondence on the line itself; as shown in Figure 12b and Figure 12c, both unit rods were used with the multi-unit rods to establish appropriate linear distances beginning at the zero point and extending to the right. In some cases, students used the unit rods only to establish the correspondence (Figure 12b), and in other cases, students used the unit rods exhaustively (Figure 12c). Figure 12d shows a coordination in which the multi-unit rod was used as only an afterthought; students constructed the target length using the unit rods (reds), marked the location of the target distance, and added the multi-unit rods (purples).

To understand whether the tutorial supported student learning, we coded the iteration at which students passed: first, second, third, or not passing any iteration. Figure 13 displays the percent of students who achieved passing performances on successive iterations of Problems Sets III and IV, respectively. The figure shows evidence of learning over problem iterations in two ways: (1) On the first iterations of Problem Sets III and IV, only a minority of students produced an adequate performance, but by the third iteration of each problem type, the large majority of students passed—89% and 95% for Problem Sets III and IV, respectively; (2) A greater percentage of students achieved passing performances for the first problem of Problem Set IV as compared to Problem Set III, even though Problem Set IV was designed to be more difficult as it involved the use of unit/multi-unit combination on the line. The finding suggests that the learning that occurred on Problem Set III was useful to students as they engaged with the more complex problems in Problem Set IV.

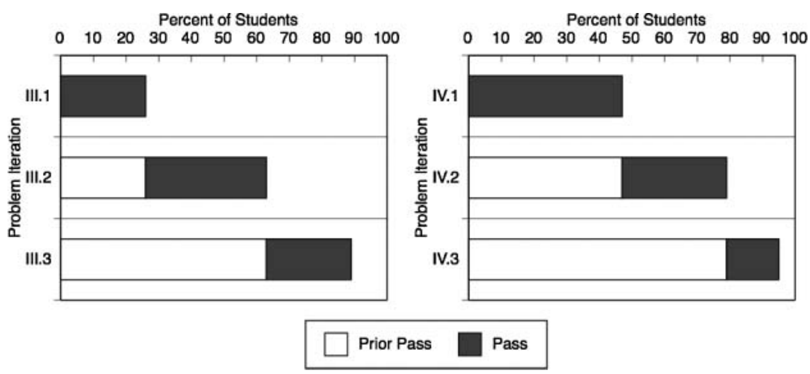


FIGURE 13 Accumulating percentages of students passing over iterations of Problem Sets III and IV.

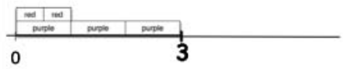
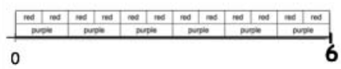
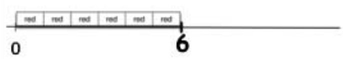
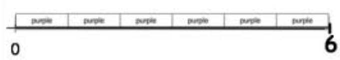
	Type of Coordination	Example for Problem III.1: "Mark 6 reds using the purples."	Proportion of Coordination Types	
			Problem Set III (n=21)	Problem Set IV (n=12)
a.	Many-to-one correspondence but recording length in terms of multi-unit instead of unit.		.10	.08
b.	Many-to-one correspondence but constructing and then recording length in terms of multi-unit.		.05	.17
c.	Inversion of unit and multi-unit correspondence (e.g., treating two multi-units as one unit)		.05	.08
d.	Use of *only* unit		.10	.08
e.	Use of *only* multi-unit		.48	.08
f.	Idiosyncratic constructions	N/A	.24	.50

FIGURE 14 Proportion of three non-passing coordination types used for iterations of Problem Sets III and IV. Unlike passing performances in which each student was represented at most once in an analysis (e.g., Figure 13 for Problem Sets III/IV), for non-passing performances, a student may have been represented not at all (passing on the first iteration), once (passing on the second iteration), twice (passing on the third iteration) or three times (not passing any iteration). As a result, we computed passing and non-passing performances differently (compare, for example, Figure 13 and Figure 14). For the analysis of non-passing performances, we computed the proportion of times a particular type of non-passing solution occurred using "type" as the unit of analysis rather than student. For Problem Set III, there were 21 no pass performances involving 13 of the 19 students (6 passed on the first iteration), and for Problem Set IV, there were 12 no pass performances involving 9 of the 19 students. For the analysis of passing performances, we simply computed percent of students who made use of a particular type of passing performance, since students could not pass any problem type more than once.

An analysis of the performance of students who received a No Pass revealed ways in which unit-to-multi-unit correspondences and their coordination with the line posed difficulties; students were producing hybrid representations constituted by the constraints of both the rods and the line. Figure 14 contains illustrations of No Pass constructions for Problem Set III. Some students determined the target length of rods with multi-units, the goal of the task, but then translated the rods into inappropriate units on the line (Figure 14a), as when the student marked a distance of 6 reds on the line with 3 purples, but labeled the endpoint of the rightmost purple with a "3" rather than a "6." Other students generated a unit-to-multi-unit correspondence on the line, but confused multi-units and units (Figure 14b), thus creating a distance of six purples rather than six reds. Other students inverted the relation of unit to multi-unit (Figure 14c), as when a student treated two purple rods as equivalent to one red rod (or two dark greens as equivalent to one light green). Still others provided no evidence of coordinating units and multi-units (Figure 14d and Figure 14e). A remainder of cases involved idiosyncratic methods of using rods in non-normative ways.

Problem Sets V and VI: Unit and Multi-Unit Relations Quantified in Rods and Line. Problem Sets V and VI were designed to be a transition in the function of the line from a recording device for rod distances to the line as a self-referential object in which the distance between 0 and 1 (or its equivalent) defined a linear unit. To support the transition, we added tick marks at irregular intervals and labeled points other than the origin, 0. The positioning of the tick marks corresponded to unit and multi-unit rod lengths. Thus Problem Sets V and VI presented students with the conceptual challenge of coordinating units and multi-units on the line (the irregularly spaced tick marks) in addition to the unit-multi-unit relations between the rods. We coded three types of passing coordinations, illustrated in Figure 15 for Problem Sets V and VI. The n in Figure 15 reflects the combined number of students who passed, whether at the first, second, or third iteration.

We coded the most common kinds of coordination to document the way students quantified the rods and the line on Problem Sets V and VI. Figure 15 displays the proportion of students using each type of coordination documented. In the most common kind of coordination, students quantified the complete target length (e.g., 5 reds) using the rods, coordinating their use of the unit and multi-unit rods with the line (Figure 15a). This strategy is similar to those passing coordinations documented on Problem Set III and IV (Figure 12), but, in addition, students (a) created the unlabeled zero-point on the line to begin their concatenation and (b) partitioned the line's multi-unit interval into two units based upon the unit rod's length.

The coordination illustrated in Figure 15b is an iterative approach to creating the hybrid representation. Students used a unit rod length to represent linear distance on the line, translating the rod through successive iterations. Sometimes they began with the unmarked zero and other times with the point marked 1 to partition the multiunit interval to locate 5. In either case, they used the unit rod to further partition the line's multi-unit interval. Figure 15c illustrates a coordination that evolved over time in the construction of a hybrid—a back and forth between the line and the rods. The locus of the quantification vacillated between rod lengths and the construction of units on the line, with activity with one form (line or rods) having implication for the other. In Figure 15c the illustration shows a student who first constructed the target length of 5 reds by placing the rods on the line (Figure 15c, Step 1). To represent the length on the line, the student unwittingly construed a multi-unit on the line as a unit based upon the inscribed tick marks (Figure 15c, Step 2); the resulting endpoint "4" conflicted with the initial rod construction of 5 unit lengths. The student subsequently revisited this conflicting information by crossing out the "4" and recording a "5" as the endpoint on the line (Figure 15c, Step 3), and then reevaluated the coordination of other numerals with the linear rod construction. The student crossed out the "3,"

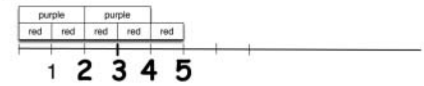
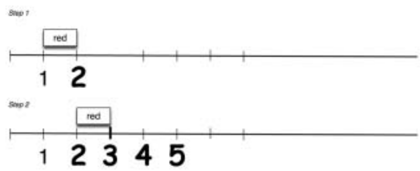
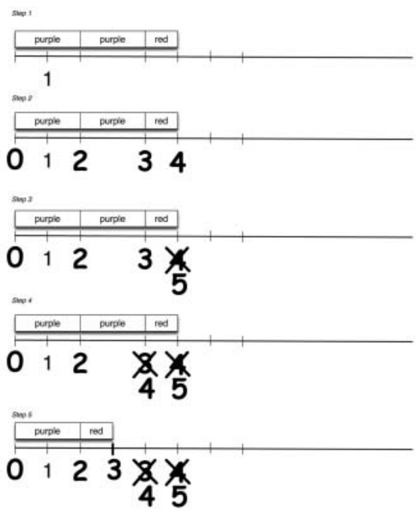
	Type of Coordination	Example for Problem V.1: "Mark where 5 reds is. You can use the red and purple rods."	Proportion of Students	
			Problem Set V (n=16)	Problem Set VI (n=15)
a.	Complete rod length used to unitize the line	Rod length constructed of either only unit rods or both multi-unit and unit rods. 	.38	.80
b.	Truncated rod length used to unitize the line		.38	.20
c.	Other: Multi-step/emergent construction		.25	.00

FIGURE 15 Proportion of students using each of three coordination types for their final (passing) iteration of Problem Sets V and VI. Of the 19 students who were administered Problem Set V, 16 students passed on one of the iterations. Of the 16 students who were administered Problem Set IV, 15 students passed on of the iterations; 3 students were not administered Problem Set VI due to time constraints.

which conflated the count of rods with linear units, and recorded a “3” and a “4” that reflected a coordination between rods and the line (Figure 15c, Step 4). In the last step (Figure 15c, Step 5), the student repositioned the rods to reflect the unit-multi-unit coordination between tick marks on the line and linear units.

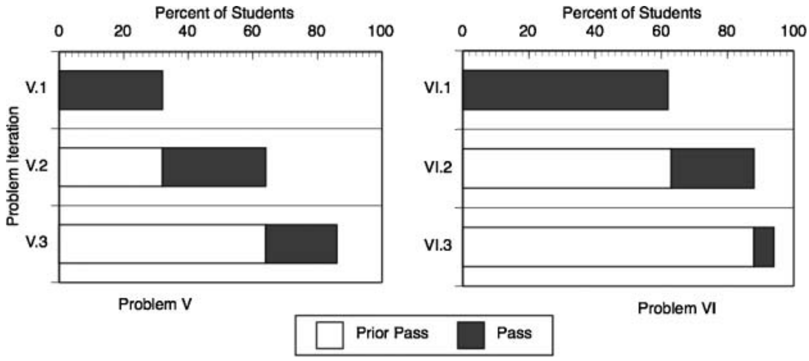


FIGURE 16 Accumulating percentages of students passing over iterations of Problem Sets V and VI. Due to time constraints, not all students received each task. One student was not administered the third iteration of Problem Set V, and three students were not administered Problem Set VI.

To understand students' trajectories of passing performances (and difficulties with the problems) through the iterations of Problem Sets V and VI, we calculated the percentage of students who achieved passing performances on successive iterations of each problem type. Figure 16 reveals that students had greater difficulty achieving passing performances with the first two problems in Problem Set V than the first two for Problem Set VI. Indeed, for Problem Set VI, a majority of students passed on the first iteration, and only two failed to pass on the second iteration.

We interpret students' difficulties with Problem Set V as resulting from the coordination problems posed by the introduction of line constraints, including unequal partitioning between tick marks and the absence of a label for 0. (This interpretation is supported by the relative success that students showed with the prior Problem Set IV, where pass rates were considerably stronger on the first and second iterations, 50% and an additional 33%, respectively.) Despite the decrement in performance on the first iteration of Problem Set V, the tutorial provided effective support; similar to the trend for Problem Sets III and IV, by the third iteration of Problem Set V, the majority of students passed the problem set, 88% and 94% for Problem Sets V and VI respectively.

The performances of students who received a No Pass revealed the challenges of coordinating quantification of rod length with quantification of the line. Figure 17 illustrates non-passing partial coordinations for Problem Sets V and VI. In the No Pass coordinations, some students constructed an accurate rod length but did not align this construction with the origin on the line (depicted in Figure 17a, Step 1); in such cases, students sometimes removed the rods after marking the endpoint (Figure 17a, Step 2). Other students treated rods as countable objects independent of unit or multi-unit length, matching a count of five rods[4] with the two interval sizes provided on the line (Figure 17b). One student did not use rods at all, counting the tick marks starting with

[4]In some cases, students failed to construct the target length in rods, instead using a truncated length to correspond rods with each of the two interval sizes on the line; students then inscribed numerals that treated each tick mark as a count irrespective of interval size.

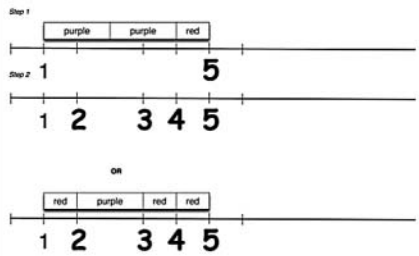
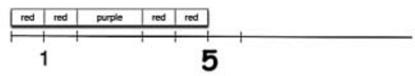
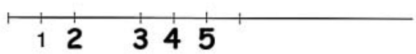
	Type of Coordination	Example for Problem V.1: "Mark where 5 reds is. You can use the red and purple rods."	Proportion of Coordination Type	
			Problem Set V (n=22)	Problem Set VI (n=9)
a.	Complete rod length is used but not coordinated with the quantification of the line.		.50	.22
b.	Rod length does not match target quantity, and line is not appropriately unitized.		.50	.67
c.	No rods		.04	.00
d.	Other	N/A	.04	.11

FIGURE 17 Proportion of four non-passing coordination types used for iterations of Problem Sets V and VI. We used coordination type rather than student as the unit of analysis for non-passing coordination analyses (see Figure 14 caption for the rationale). For Problem Set V, there were 22 cases involving 13 of the 19 students. For Problem Set VI, there were 12 cases involving 6 of the 16 students; three students were not administered Problem Set VI due to time constraints.

1 up to 5 (Figure 17c), and two other students provided idiosyncratic interpretations of the task ("other" in Figure 17d).

Problem VII and VIII: Unit-to-Multi-Unit Relationship Quantified on Line Only. Problem Sets VII and VIII marked the transition to the use of a number line in which the units were defined internal to the line itself, with no extrinsic links to Cuisenaire Rods. For these problems, the unit distance was defined by the 0 to 1 interval (Problem Set VII) or its equivalent (Problem Set VIII). The passing coordinations revealed ways that students used the unit or multiunit defined on the line as a resource to identify the target point. We observed two distinct types of coordinations depicted in Figure 18 and illustrated for problem VII.1. In the first type, students used an improvised tool, such as pinched fingers or a pen cap, to measure the distance between 0 and 1, and then iterated

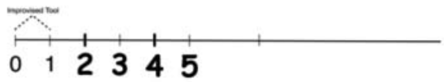
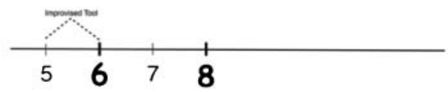
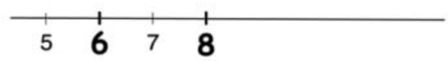
	Type of Coordination	Example for Problem VII.1: "Mark where 5 reds is." and Problem VIII.1: "Mark where 8 is"	Proportion of Students	
			Problem Set VII (n=14)	Problem Set VIII (n=15)
a.	Linear unit encoded in improvised tool to partition multi-unit.	Problem VII.1  Problem VIII.1 	.57	.27
b.	Linear unit visually estimated to partition multi-unit	Problem VII.1  Problem VIII.1 	.43	.73

FIGURE 18 Proportion of students using each of two coordination types for their final (passing) iteration of Problem Sets VII and VIII. For passing coordinations, the *n* reflects students who passed at any of the three iterations for each problem set, respectively. For Problem Set VII, this included 14 of 15 students (4 students were not administered the problem set due to time constraints). For Problem Set VIII, this includes 15 of 15 students (missing 4 children due to time constraints).

this measure (Figure 18a). The second type involved the use of a visual estimate and subdividing multi-unit intervals in order to create equal partitioning (Figure 18b).

Figure 19 displays the proportion of students who achieved passing performances on successive iterations of Problem Sets VII and VIII. Results indicate that Problem Sets I through VI prepared students well for the transition to the new function of the line. Of the 15 students who were administered Problem VII, nine (60%) passed at the first iteration, 12 (87%) passed by the second iteration, and 14 (93%) by the third iteration. On Problem Set VIII, the passing performances occurred at a more rapid rate. Twelve students (87%) passed at the first iteration, and the remaining students passed at the second iteration. (Because all students passed by the second iteration, the third iteration was not used for any students.)

Students who did not pass an iteration of Problem Set VII sometimes produced partial coordination of units and multi-units as defined on the line, and at other times did not coordinate these at all. The partial coordinations varied, but all had two common properties: (1) the 0 to 1 interval was not used consistently to define unit lengths; (2) the unit length varied on the line, with the multi-unit often treated as a unit distance. Figure 20 illustrates the variety of non-passing

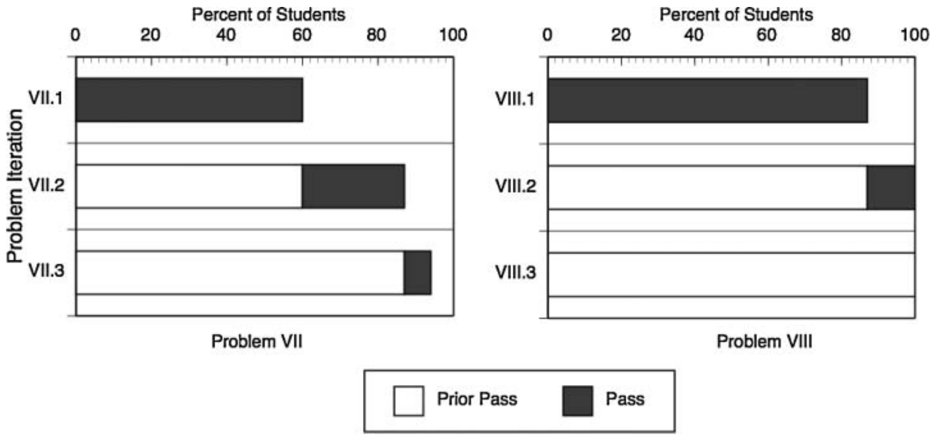


FIGURE 19 Accumulating percentages of students passing over iterations of Problem Sets VII and VIII.

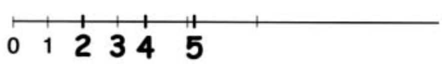
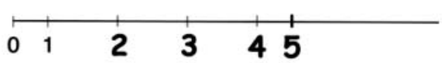
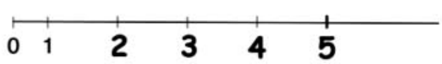
	Type of Coordination	Example for Problem VII.1: "Mark where 5 is."	Proportion of Students: Problem Set VII (n=9)	Proportion of Students: Problem Set VIII (n=2)
a.	Attempt to measure the unit distance but imprecise		.11	.50
b.	Inconsistent application of unit distance		.22	.50
c.	Consistent application of unit distance, but not coordinated with 0 to 1		.67	.00

FIGURE 20 Proportion of three non-passing coordination types used for iterations of Problem Sets VII and VIII. We used coordination type rather than student as the unit of analysis for non-passing coordination analyses (see Figure 14 caption for an explanation). For Problem Set VII, there are 9 cases involving 6 of the 15 students. For Problem Set VIII, there are 2 cases involving 2 of the 15 students. For both Problem Sets, 4 students were not administered either due to time constraints.

performances. One student attempted to measure the unit distance but was imprecise (Figure 20a). Other students were inconsistent with applying the unit distance to the task of locating the target distance on the line (Figure 20b). Some showed a consistent unit distance, but one that was not coordinated with the distance from 0 to 1 (Figure 20c). These performances illustrate the challenges of applying prior agreements to new tasks that did not have linear measurement tools available.

A Student's Trajectory through Problems III–VIII

The prior analyses of pre- to posttest gains, the role of agreement use in gains, and the character of student solutions in the Session 1 problems reveal how participation in the tutorial supported student learning. In this section, we report a case study to provide insight into students' trajectories as they worked through a sequence of tutorial problems, drawing on agreements to mediate their constructions. Our case student is Dustin, and we focus on his efforts to coordinate placements of points on the line with the tutor as he moved through Problem Sets III–VIII. We had two guiding purposes for this analysis of the dynamic aspects of the tutorial sessions. The first was to highlight continuities and discontinuities in Dustin's trajectory of solutions. The second was to understand the way Dustin's trajectory of solution approaches was interwoven with discrepant solutions, and the ways that tutor and student invoked agreements in their efforts to reconcile discrepancies.

We chose Dustin as our case for two reasons. First, Dustin followed a typical trajectory as he progressed through the tutorial problems, although his gain from pre- to posttest was at the upper end of the distribution. Second, Dustin showed gains on three pre- to post-assessment items that were similar to Problem Sets VII and VIII, items requiring a coordination of numerical and linear units independent of the rods. Figure 21 shows the percent of students who passed these three assessment items on pre- and posttest for the tutorial and control groups. Dustin's performance, like most students in the tutorial group, showed marked change. At pretest on the three items, Dustin showed no evidence of coordinating numerical and linear units. For instance, on one problem, a number line was shown with 5 and 6 marked on it, and students were asked to place 8 on the line. Dustin placed the 8 at the tick mark that should have been the location for 9. On the posttest, like many in his tutorial cohort, he successfully coordinated numerical and linear units in positioning the numbers for each of these three numerical-linear unit coordination problems.

Figure 22 contains a profile of Dustin's trajectory from Problem Sets III–VIII on the x-axis, showing which problem iteration he passed on the y-axis. Like most tutorial students, Dustin ultimately achieved a pass on every problem type, passing by the third iteration on each. Further, like many, Dustin showed greater difficulty passing iterations on the first three problem sets (Problem Sets III, IV, and V) than the last three.

Problem Sets III and IV. As indicated in Figure 22, Dustin was presented with all three iterations of Problem Set III, passing on the third (III.3). The character of Dustin's efforts, his interaction with the tutor, and the creation and use of agreements illustrates the dynamics of the tutorial procedure as he shifted in his quantification of the rods, the line, and his coordination of them.

As depicted in Figure 23, for each iteration of Problem Set III, Dustin started his construction on the line at the zero point and proceeded to the right, consistent with the order agreement that

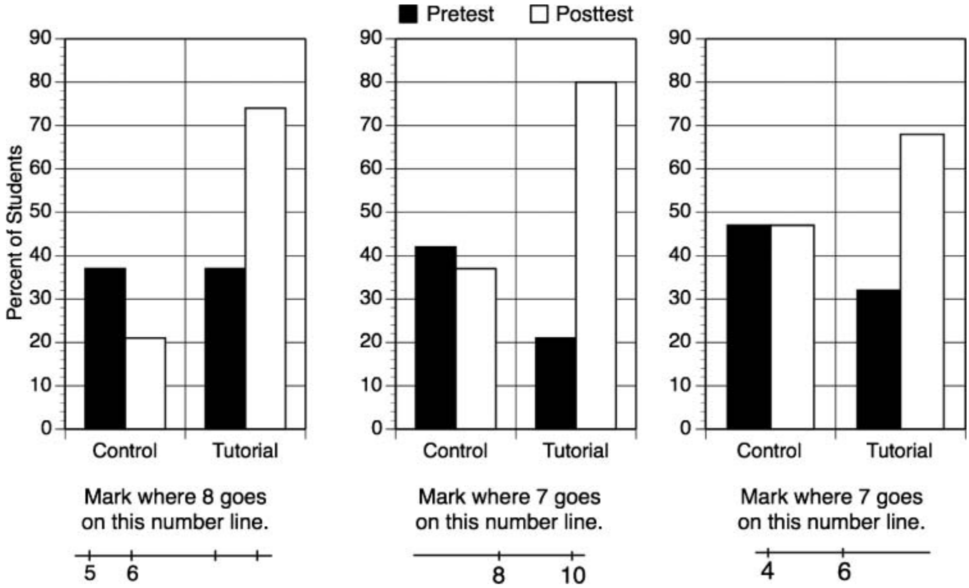


FIGURE 21 Percentage of pre- and posttest passing performances for control and tutorial groups on problems requiring a coordination of numerical and linear units.

had been established in the introductory Problem Set I. But for the first two iterations, Dustin's approach to quantifying unit-to-multi-unit relations with the rods was inconsistent with the skip counting agreement, and the inconsistency created a context for the tutor to support Dustin's reflection on the tutor-student agreements. When presented with the first iteration (III.1)—"Mark where 6 red rods is using purple rods"—Dustin concatenated six (instead of 3) purples, and

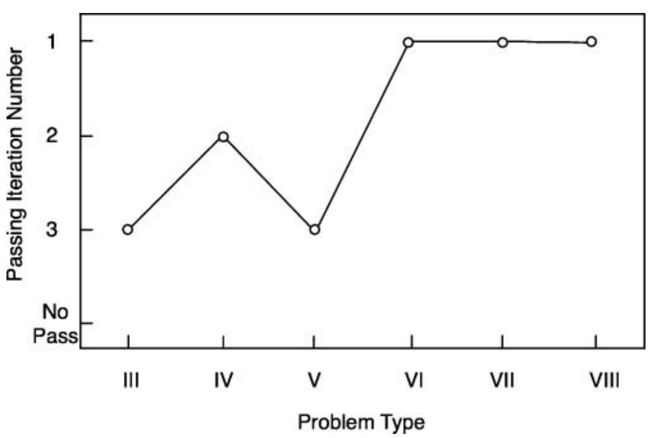


FIGURE 22 Dustin's passing trajectory over iterations of Problem Sets III through VIII.

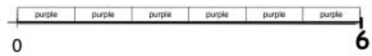
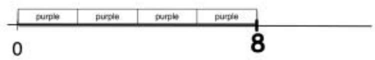
Problem Iteration	Pass?	Problem and Student's Solution	Agreement Usage
III.1	No Pass	Mark where 6 reds is using purple rods. 	Tutor introduced the Skip Counting Agreement.
III.2	No Pass	Mark where 4 reds is using purple rods. 	Tutor referenced the Skip Counting Agreement.
III.3	**Pass**	Mark where 8 reds is using purple rods. 	

FIGURE 23 Dustin's performance on his three iterations of Problem Set III.

marked the end point of the purples with the numeral "6," twice the length of the tutor's (a solution type represented in Figure 14e). When the tutor's line was overlaid on Dustin's, Dustin commented on the tutor's line noting, "You used the red one in twos" (tutor's line was labeled: 0, 2, 4, 6). In the ensuing interaction, the tutor asked Dustin what he did, and Dustin indicated that he used purples. In turn, the tutor explained her approach, indicating that she "skip counted," and referred to the 2:1 relation between reds and purples, saying "this is like 2 reds, 4 reds, 6 reds." Towards the end of the interaction, the tutor asked, "So what do you think we should do next time to make sure that our 6s are in the same place?" and Dustin responded, "Use 6 reds." The tutor went on to make reference to skip counting and its application to reds and purples. In the end, the tutor expressed the Skip Counting Agreement as a means to (a) resolve the tutor and student's lack of coordination and (b) improve placements on the next problem.

Because of Dustin's No Pass for Problem III.1, the tutor drew the Problem III.2 problem card, which stated, "Mark 4 reds using the purple rods." In their initial solutions, both Dustin and the tutor placed the tick mark at the same place, each labeling the position appropriately as "4." But Dustin used only the unit rod (reds) to mark the target length (a solution he offered to solve the discrepant placements in the prior iteration, Problem III.1). To support Dustin's construction of a unit-multi-unit coordination, the tutor once again pointed to the Skip Counting Agreement to produce a length of 4 reds using only the purples, emphasizing ways of conceptualizing an interpretation of a length of reds (units) in terms of purples (multi-units). The tutor drew problem card III.3—"Mark where 8 red rods is using purple rods." Dustin coordinated unit to multi-units off the line, putting 2 reds on top of 1 purple rod, and then lined up successive multi-unit rods independently of the unit rod (analogous to Figure 14a). In his explanation, Dustin referred explicitly to the unit-multi-unit coordination, "because two [reds] fit on a purple."

With Problem Set III successfully completed, the tutor drew the card for the first iteration of Problem Set IV—"Mark where 3 light greens is using both light green and dark green rods." Dustin concatenated six light green rods on the line and marked the endpoint as "3," producing a length twice that of the tutor's (Figure 24, IV.1). When the tutor's number line was overlaid, the

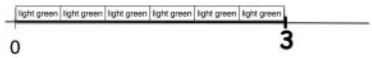
Problem Iteration	Pass?	Problem and Student's Solution	Agreement Usage
IV.1	No Pass	Mark where 3 light greens is using both light green and dark green rods. 	Tutor introduced the Every Number Has a Place Agreement.
IV.2	**Pass**	Mark where 5 light greens is using both light green and dark green rods. 	The Every Number Has a Place Agreement is brought up again.

FIGURE 24 Dustin's performance on his two iterations of Problem Set IV.

tutor asked Dustin if he used any agreements, to which he replied that he had not. In an effort to support Dustin's learning, the tutor reflected, "It looks like you did," showing how Dustin's construction was consistent with the Order Agreement. The tutor then referred to the problem card that asked for a length of 3 light greens, and pointed out that Dustin actually marked where 3 dark greens was instead of 3 light greens. Like Problem Set III, the tutor explained her solution, emphasizing the unit-to-multi-unit ratio between light and dark greens that she "discovered" while marking her number line as well as the directions on the problem card. Consistent with the protocol, the tutor introduced a new agreement—the *Every Number Has a Place but Doesn't Have to Be Shown* Agreement. To do this, the tutor noted to Dustin that he had marked only 0 and 3 on the number line. Using Dustin's construction, she asked, "Where would the 1 go if it were on there? What about the 2?" She emphasized that they both knew this even if it had not been written, creating an opening for the new agreement.

With the failure to reach the same placement for Problem IV.1, the tutor drew problem card IV.2, which stated, "Mark where 5 light greens is using the light green and dark green rods." Dustin concatenated two dark greens with one light green off the line, and then placed them on the line using zero as the start point (Figure 24, IV.2). Following Dustin's explanation, the tutor modeled the use of the new agreement by commenting that each of them omitted the "1" from their final number line, but that they each knew it had been there because every number has a place.

Problem Sets V and VI. Problem Sets V and VI provided a transition between the line as a recording device to that of a self-referential representation. Dustin worked on all three iterations of Problem Set V, but then passed Problem Set VI on the first iteration. Dustin's constructions for these two problem sets are shown in Figure 25 and Figure 26.

When presented with Problem V.1—"Mark where 5 reds is"—Dustin began by concatenating rods directly on the line. After concatenating two reds starting at the unmarked zero point, he laid a purple on the multi-unit interval (see Figure 25, V.1), completing the line with reds. Dustin then incorrectly treated the multi-unit purple length as if it were a unit length and marked the equivalent of a length of 6 reds with the label "5." When Dustin compared his own construction with that of the tutor's through the overlay procedure, he noted the discrepancy. Dustin indicated that he used "2 reds and then a purple and then a red and a red" as he pointed to each of the

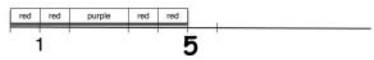
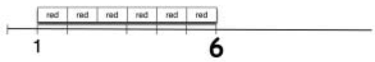
Problem Iteration	Pass?	Problem and Student's Solution	Agreement Usage
V.1	No Pass	Mark where 5 reds is. 	Tutor referenced the Unit Distance Agreement.
V.2	No Pass	Mark where 6 reds is. 	Tutor referenced the Every Number Has a Place Agreement.
V.3	**Pass**	Mark where 4 reds is. 	

FIGURE 25 Dustin's performance on his three iterations of Problem Set V.

intervals on the line. The tutor then graciously indicated that she almost did the same thing, but explained that as she was constructing her line she remembered the agreement that the distance between counting numbers had to be the same. She showed that if the distance from 0 to 1 is a red and 1 to 2 is a red, then another red must indicate where "3" should be. Toward the end of the interaction, the tutor asked what they could do next time to ensure similar point placement. Dustin responded, "Use the same space?"

For iteration V.2, the game card stated, "Mark where 6 reds is." Dustin constructed the targeted rod length, though he aligned the rod length with "1," not conceptualizing the "1" as representing one linear unit from the origin (see Figure 25, V.2). As a result, Dustin wrote "6" at the 7 position. When the two number lines were overlaid, Dustin shook his head upon seeing that they did not achieve the same point placement. Using the tutor's number line overlaid on his own, Dustin pointed to the 1 on the number line and observed, "I used [the] one right here because I forgot." The tutor clarified with Dustin that he treated the "1" as if it had been 0. In the interaction, the tutor stated, "But it looks like we did the same thing. We knew that the space here was too big, and it had to be cut in half." Because Dustin did not write in any numbers besides the 6, the

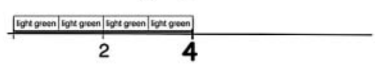
Problem Iteration	Pass?	Problem and Student's Solution	Agreement Usage
VI.1	**Pass**	Mark where 4 light greens is. 	

FIGURE 26 Dustin's performance on his only iterations of Problem Set VI.

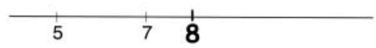
Problem Iteration	Pass?	Problem and Student's Solution	Agreement Usage
VII.1	**Pass**	Mark where 5 is on the number line.  0 1 **5**	Student referenced the Unit Distance Agreement
VIII.1	**Pass**	Mark where 8 is on the number line.  5 7 **8**	Tutor referenced the Unit Distance and Every Number Has a Place Agreements

FIGURE 27 Dustin's performance on his only iterations of Problem Sets VII and VIII.

tutor brought up the Every Number Has a Place Agreement, noting that Dustin could write the numbers between 0 and 5 as a way to check his own work.

The tutor drew the card for Problem V.3—"Mark where 4 reds is." Dustin carefully lined up three red rods, coordinating the "1" on the number line with the endpoint for the first linear unit, and coordinating the multi-unit on the line between "1" and the next tick mark with two red rods. He wrote in only the "4" to mark the endpoint of the four red rods. Thus unlike the previous problem, Dustin did not use the complete correct rod length. Instead, he coordinated physical rods with linear distances marked on the line to construct an appropriate coordination of the rods and the line in his labeling of 4 reds.

For Problem VI.1, Dustin made use of the agreements and ideas expressed in discussions of the prior problems, even though the rods shifted in color and length. The problem card stated "Mark where 4 light greens is." Dustin constructed 4 light green rods and aligned it with the 0 tick mark (Figure 26, VI.1).

Problem Sets VII and VIII. On Problem Sets VII and VIII, Dustin passed on the first iteration of each, displaying a coordination of unit and multi-unit linear distances. On problem VII.1, Dustin correctly located the 5 on the number line without the use of rods, instead using pinched fingers to iterate the unit (Figure 27, VII.1). The tutor asked about his use of agreements, and Dustin responded that he used the agreement about the same distance between counting numbers. Dustin extended this approach to the more complex coordination on Problem VIII.1, where student and tutor were presented with a number line with only the 5 and 7 marked and required to place the number 8. In Dustin's approach, he used pinched fingers to divide the multi-unit of 2 (the 5 to 7 interval) into two unit intervals. Without making a mark, he gestured to the (unmarked) interval from 5 to 6, and then to the (unmarked) interval from 6 to 7, before extending his unit iteration to 8 (Figure 27, VIII.1). When the tutor asked how he placed the 8, Dustin responded that he used his fingers, demonstrating his strategy. The tutor reflected that she also thought about that, calling upon the agreement about the same distance between counting numbers and that every number, like 6, has a place but does not have to be shown.

Dustin's performance and use of agreements on the final, more complex problems reflects Dustin's developing strength in coordinating numerical and linear units, a shift reflected in his pre- to posttest change score and consistent with trajectories of most other students in the tutorial cohort.

DISCUSSION

Our analyses showed that a communication game was an effective tutorial strategy for supporting students' development of important mathematical ideas related to the number line. In our discussion, we consider evidence of the efficacy of the tutorial, the dynamics of tutorial interactions that supported learning, and microgenetic processes in students' constructions with attention to the curricular/pedagogical and cognitive frameworks that guided our tutorial design.

The Efficacy of the Tutorial

Several of our analyses revealed that the tutorial led to student learning about important mathematical ideas. Perhaps the clearest evidence comes from our analysis of the pre- to posttest gains of the tutorial group. The analysis revealed that tutorial students improved in their overall performance from pre- to posttest, and the effect size was large, approaching two standard deviations.

In our research design to evaluate the efficacy of the tutorial, we were careful to control for two threats to validity. The first threat was a regression to the mean at posttest. Recall that the students who participated in the study performed at the lower half of their classroom populations (at pretest). Thus at posttest these students might appear to develop new understandings across testing sessions when the gain was (to some extent) an artifact of initial measurement error. The second threat was practice with the assessment items: Since we used the same assessment for pre- and posttest, improvement at posttest could be attributed to prior experience with the items at pretest.

To address these threats in our design, we randomly assigned students matched for both pretest score and classroom to the tutorial and control groups. Our analysis revealed that the improvement of the control students was not comparable to the improvement of tutorial students, indicating that tutorial students' gains were due to participation in the tutorial procedure and not due to a statistical artifact (regression to the mean) or repeated testing.

Another source of evidence for the tutorial's efficacy was analysis of students' performance across problem iterations in the tutorial. For most of the problems, many students did not solve the first iteration adequately, but by the third iteration, all or almost all students passed. One critique of this evidence could be that students were simply memorizing solutions on the earlier iterations and using them on subsequent ones. But again we purposely controlled for this potential threat to validity by shifting the values of the target numbers and the numbers across problem iterations, so improvements could not result from mere memorization. Further, in Session 1, problem sets were designed as related pairs (e.g., Problem Sets III and IV), and the second set was more complex than the first. Students often solved the problems in the second set more quickly (with fewer iterations) than the first set, again corroborating a learning effect of the tutorial procedure.

Dynamics of Tutorial Interactions that Supported Learning

Guided by our curricular/pedagogical model, we designed the tutorial with features that afforded the emergence of a supportive environment for learning. These features included: Cuisenaire Rods that served as physical magnitudes to be recorded on an open number line; a sequencing of problem sets in Session 1 (from recording functions through self-referential functions) and Session 2 (from negative number to absolute value/symmetry); a progressive development of planned agreements that afforded student generative use of number line principles to mediate their number line constructions and adjudicate discrepant solutions to tutorial problems; wrap-up sessions to support and consolidate learning; iterations within problem sets to provide students multiple opportunities to work through the ideas supported by the tutorial. Together, these design elements were intended to afford emergent activities that would support generative number line understandings. A factorial design in which we systematically controlled or varied design features was not feasible, nor was it conceptually warranted given our focus on the emergence of a supportive learning environment (Yin, 2009). Our central concern was analysis of the interplay of the ways that students and tutors drew on design elements like Cuisenaire Rods, tasks, tutorial strategies, and agreements as they progressed through the tutorial.

We focused on agreement use as an important indicator of the ways that students were learning to solve problems and build understandings of core mathematical ideas. For this analysis, we treated students' performance in the wrap-up sessions as a measure of students' mindful use of agreements in approaching number line problems. Students varied in the extent that they used the agreements to organize their problem solving, and this variation enabled us to corroborate whether reflective agreement use supported learning. We found that mindful use of agreements differentiated students who gained more and less from the tutorial. In Session 1, agreement use was related to the understanding of order, unit, and origin as students moved from the use of the line as a recording device to the use of the line as an autonomous representational object. In Session 2, agreement use was related to the understanding of entailments on the number line itself, as students constructed negative numbers, order relations across negatives and positives, and reflected on problems involving symmetry as a means of solving problems. We interpret the positive effects of the tutorial sessions as the result of the complex of design elements that were mediated through the construction and use of agreements.

Microgenesis of Number Line Representations in Students' Solutions to Tutorial Problems

Several of our analyses focused on the process whereby students constructed number line representations over the course of the Session 1 tutorial. Our framework provides an account of the construction of modeling/recording functions (Figure 1) and self-referential functions of the number line (Figure 2), and we use it here to reflect on the ways that students were coordinating Cuisenaire Rods and lines more and less successfully, as well as ways that students' iterative engagement with the number line across problem sets supported their learning.

On the open number line modeling/recording problems (Problem Sets III, IV), we documented students' quantifications of the rods, quantifications of the line, and their coordination of these in the construction of hybrid representations. (See the three strands of activity depicted in Figure 1.)

Most students' treatments of the rods were eventually informed by their conceptualization of the line; students concatenated the rods end-to-end, anticipating that the arrangement of rods needed to be linear to be represented on the line. Reciprocally, most students eventually conceptualized the line in terms of the rods; for example, the zero point on the line became an origin for a measure of the rods in microgenetic constructions. Although the line and the rods were resources in virtually all students' construction of hybrid representations on the modeling/recording problems, many students did not initially produce adequate representations for the first iteration of Problem Set III (Problem III.1, Figure 13). When conceptualizing the rods in terms of the line, for example, many students did not differentiate multi-units and units. When interpreting the line in terms of the rods, a few students did not conceptualize the distance on the line as a distance from the zero point (even though the zero point was marked). The microgenesis of students' coordinated construction of rod and line units improved over the iterations of the modeling/recording problems. Almost all students achieved adequate coordinations by the third iterations of Problems III and IV, although there was variation in the sophistication of successful strategies.

The two transitional problem sets (V and VI) were designed to maintain continuity with the prior open number line recording problems involving unit/multi-unit rod relations (Problem Sets III and IV), while providing a segue into problems in which the number line was used as an autonomous, self-referential object (Problem Sets VII and VIII). In Problem Sets V and VI, we added irregularly spaced tick marks and changed the single number represented on the line from 0 (in Problem Sets III and IV) to 1 (Problem Set V) or 2 (Problem Set VI) in order to support students' principled use of agreements and rods despite non-canonical number lines. As we expected, these changes created challenges for students, although their prior learning provided a foothold. When conceptualizing the rods, students were often initially influenced by the irregularly positioned tick marks and did not organize their solutions by the unit distance principle; when zero was not shown, some students treated another number as the origin. But most students worked through these challenges, and by the end of the third iterations almost all students achieved adequate constructions.

In the final problems in Session 1 (Problem Sets VII and VIII), we specified the positions of two numbers on the line, and thus the line became a representation in which all (real) numbers had a place without any reference to rods at all. For these problems, the character of students' hybrid constructions shifted to a coordination of numerical units (a count of positions of points on the line) with linear units (congruent intervals). Students' prior work with the tutorial problems with rods and agreements appeared to prepare them well for the microgenetic construction of these representations in which Cuisenaire Rods played no discernable role. Though most tutorial students did not pass this type of item on the pretest, most solved the problem in the first iterations of the tutorial and appeared to draw on what they had learned from working on the modeling and transition problems using the associated agreements.

CONCLUDING REMARKS

When students are presented with number line representations in school, they typically do not explore the generative principles underlying the geometric and numeric properties of the line, and the varied normative conventions for interpreting number line representations. The consequence for students is that many leave the elementary grades with shallow understandings that are not generative across number line representations and problems; indeed, few students in this study showed initial evidence of understanding fundamental principles of integers on the line, principles such as linear unit, multi-unit, and absolute value. The lack of focus on generative understanding of the line is unfortunate, especially in light of the frequent use of the number line in secondary mathematics and beyond. Our findings—the efficacy of the tutorial, students' learning trajectories, mediating effects of agreement use, and the microgenesis of representations—provided empirical corroboration for the utility of the approach for supporting student learning as well as illuminating the conceptual and interactional processes that support learning trajectories for hard-to-learn, hard-to-teach ideas.

The framework and findings from this study are a core resource for our current research and development work in the Learning Mathematics through Representations project. Building on the tutorial study and other research, we are designing a curriculum unit on integers and fractions, using the number line as the principal representational context (e.g., Saxe et al., 2009; Saxe et al., 2009; Saxe et al., 2007). As we scale up to the level of the classroom, we are adapting the sequence of tutorial problem sets as a "problems of the day" lesson format that engages students with challenges requiring the class, over time, to construct and apply "Number Line Principles and Definitions." Our pilot studies are yielding very promising evidence that the approach can support students' rich understandings of integers and fractions on the number line. We believe that the principles-based approach to mathematics teaching and learning will eventually have utility in other domains and at other grade levels.

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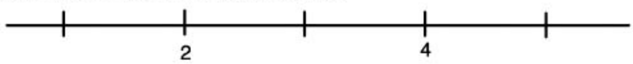
APPENDIX

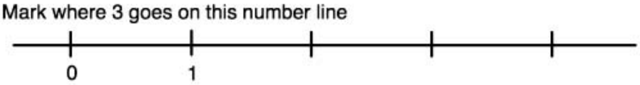
Assessment Items Used on the Pre- and Posttests

Name: ____________________

1. Mark where 1 goes on this number line



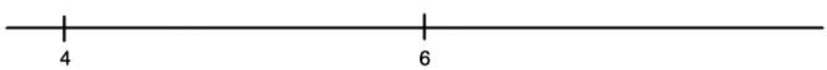
2. Mark where 3 goes on this number line



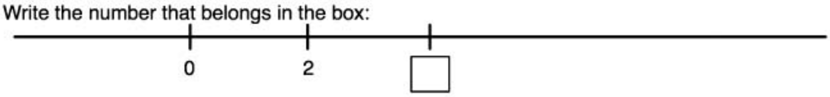
3. Mark where 8 goes on this number line

4. Mark where 7 goes on this number line

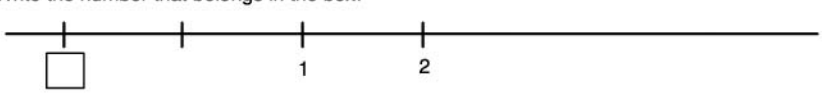
5. Mark where 7 goes on this number line



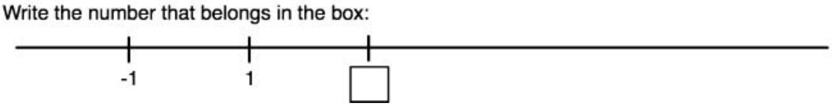
6. Write the number that belongs in the box:



7. Write the number that belongs in the box:

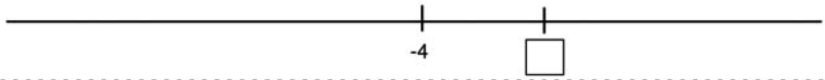


8. Write the number that belongs in the box:

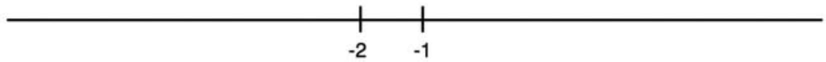


Appendix (continued)

9. Write a number that belongs in the box:



10. Mark the place on the number line that is 3 less than -2.



11. Write two numbers greater than -2. ____________ ____________

12. Write these numbers from least to greatest: -4 0 1 ________ ________ ________

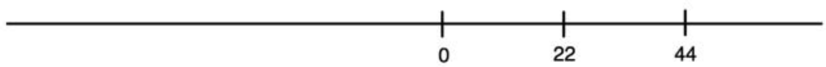
13. Write these numbers from least to greatest: -6 -5 -9 ________ ________ ________

Look at these numbers: 4, 3, -5

14. Write the number that is closest to 0: __________

15. Write the number that is furthest from 0: _________

16. Mark -22 on the number line.



17. Mark 0 on the number line.