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Source: *The Journal of the Learning Sciences*, Vol. 11, No. 2/3, Analyzing Tools: Perspectives on the Role of Designed Artifacts in Mathematics Learning (2002), pp. 275-300
Published by: Taylor & Francis, Ltd.
Stable URL: https://www.jstor.org/stable/1466665
Accessed: 22-12-2019 16:43 UTC

THE JOURNAL OF THE LEARNING SCIENCES, *11*(2&3), 275–300

Children's Developing Mathematics in Collective Practices: A Framework for Analysis

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This article presents a cultural–developmental framework for the analysis of children's mathematics in collective practices and illustrates the heuristic value of the framework through the analysis of videotaped episodes drawn from a middle-school classroom. The framework is presented in 2 related parts. The first targets the children's emerging mathematical goals in collective practices, with a particular focus on the complex role that artifacts play in children's emerging goals. The second part focuses on children's developing mathematics that takes form in their goal-directed activities: (a) Microgenetic analyses concern the process whereby children structure cultural forms like artifacts to serve particular functions as they accomplish emerging mathematical goals; (b) sociogenetic analyses concern the spread or travel of mathematical forms and associated functions within a community of individuals; and (c) ontogenetic analyses concern the interplay between the forms that children use and the functions that they serve over the course of children's development. The analyses of the classroom episodes points to the promise (and limitations) of the framework as a method for furthering our understanding of the interplay between social and developmental processes in children's mathematics.

A growing body of classroom-based research in mathematics education is concerned with understanding the role of artifacts in processes of teaching and learning. In her classroom-based research, Ball (1993), for example, pointed to the way in which the use of one kind of artifact—the area of geometrical shapes—provides a representational context to explore properties of fractions. Lampert (1986) pointed to the utility of currency as support for teaching and learning about

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multidigit arithmetic. Cobb and his associates (e.g., Cobb, Wood, Yackel, & McNeal, 1992) analyzed the way artifacts, like educational manipulatives, support children's developing understanding of numeration and place value. And Sfard (2002/this issue) targeted the dynamics of meaning making as children structure communications with one another relative to particular artifacts. Across these approaches to the analysis of artifacts, we find that authors point to ways in which artifacts both enable and constrain opportunities for teaching and learning. In this article, I build on treatments of artifacts and children's mathematics in an analysis of teaching and learning in episodes drawn from a middle-school classroom, Kay McClain's and Paul Cobb's stats project class.

The stats project class is a component of Cobb and colleagues' research and curriculum development effort. The sequence of instructional activities that Cobb and colleagues have crafted support student and teacher integrated exploration of what are often segregated topics in middle-school statistics, topics like central tendency, proportional reasoning, and data representation. A key ingredient of the stats project class activities was the use of computer "minitools," a software program that afforded students and teacher a means to structure inquiry about data sets through the generation and modification of statistical displays. As teacher, Kay McClain drew students into inquiry involving the software; students supported and took issue with one another's conjectures and arguments in what appear to be pedagogically significant ways.

Like other contributors to this volume I consider the role that artifacts, such as software-linked statistical displays, play in processes of teaching and learning in collective practices like that of the stats project class. However, I take a different analytic turn. I introduce a framework that has taken form in my prior work on the interplay between cultural and developmental processes in children's mathematics (Saxe, 1991, 1999; Saxe, Dawson, Fall, & Howard, 1996; Saxe, Gearhart, & Seltzer, 1999; Saxe, Guberman, & Gearhart, 1987). A fundamental assumption of the approach is that novel cognitive developments are rooted in goal-directed activities. I show how the framework enables analyses of the complex role that artifacts play in children's emerging mathematical goals as well as analyses of children's developing mathematics as they structure and accomplish these goals.

AN OVERVIEW OF THE ANALYTIC APPROACH

Collective practices are recurring, socially organized activities that involve the engagement of multiple participants. They are pervasive in children's and adults' everyday lives. They include play of games like Monopoly or chess as well as less formal activities like making store purchases or selling lemonade. In collective practices, joint tasks are accomplished—the completion of a game or the sale of goods—through the interrelated activities of individuals. In such joint accomplishments, individual and collective activities are reciprocally related. Individual activities are constitutive of collective practices. At the same time, the joint activity of the collective gives shape and purpose to individuals' goal-directed activities.[1] I use two related strands of analysis in an effort to understand the role of artifacts in developmental processes in the episodes drawn from the stats project class.

The first strand is a frame for analyzing children's emerging mathematical goals in collective practices. I point to the way in which goals emerge in relation to the structure of activities, social interactions, artifacts, and the prior understandings that students bring to the stats project class.

The second strand is a frame for analyzing the development of children's mathematics. I consider three broad kinds of developmental processes. Each of these involves an interplay between cultural or material forms and the functions that they afford in individual activity. The first involves the short term process whereby individuals structure forms into means to accomplish goals in activity (microgenesis). The second involves the spread in use of particular forms as means for structuring and accomplishing goals in a community of individuals (sociogenesis). The third involves the interplay over the course of individual development between the use of forms and functions that they serve in structuring and accomplishing goals (ontogenesis). Using the episodes from the stats project class, I illustrate these kinds of developmental processes and show how their interrelations are central to understanding children's mathematics.

FIRST STRAND: EMERGING MATHEMATICAL GOALS IN THE DYNAMICS OF CLASSROOM PRACTICES

The view that children's mathematics is a product of their own goal-directed sense-making activities is widely held by developmental psychologists (Piaget, 1970; Vygotsky, 1978, 1986). Despite the importance of the idea, we have few efforts at systematic analysis of the processes whereby mathematical goals take form and are accomplished in collective practices. This is a challenging analytic task, one that must be at the heart of a treatment of learning and development.

In this section, I sketch a frame for understanding children's emerging goals, drawing on the stats project class episodes related to the *Batteries* [1–49] and *AIDS* [50–153] episodes. I begin the effort by identifying four dimensions of children's activities, considering the way in which each is implicated in children's emerging

[1]Prior analyses using the framework have focused on practices of candy selling by children in Brazil's northeast (Saxe, 1991), economic exchange by individuals in a remote cultural group in Papua New Guinea (Saxe, 1982), mother–child engagement with activities involving numbers in middle- and working-class communities in the United States (Saxe, Guberman, & Gearhart, 1987), and children's play of a board game involving mathematics (Saxe & Guberman, 1998).

goals. The dimensions include activity structures, social interactions, artifacts, and the mathematical understandings that children bring to collective practices.

Activity Structures

Collective practices are routine social activities constituted by patterned ways that individuals participate with one another—a patterning that I will refer to as an "activity structure." Key properties of activity structures include routine phases or cycles of activity, norms and sometimes explicit rules for behavior, and emerging role relations between participants.

The idea of an activity structure helps us interpret the goals that emerge for individuals in collective practices. For example, the goals that emerge for individuals in chess are different from those that emerge for a customer and cashier completing a transaction at a store. In chess, the activity structure typically involves two opponents taking turns through the opening, the middle, and the end of the game, who are constrained by chess-specific rules and norms for etiquette (e.g., time spent on a move). Such goals are quite different from those that emerge in making a purchase at a store, in which the activity structure involves selection of commodities, offering payment, and return of change, if required.

Collective practices and their associated activity structures are historically situated, taking shape and gradually shifting in their organization in a complex of economic, social, and political circumstances, which themselves are in motion. In the case of the stats project class, we find evidence of an activity structure related to mathematics instruction that has emerged over the history of schooling in Western societies. It is a structure that is aligned well with the movement toward a particular brand of educational reform. In contrast to "traditional" practices that favor teacher exposition and student drill, the activity structures valued in reform-oriented classrooms support student inquiry and understanding. In such structures, the teacher takes on the role of facilitator, working to problematize student contributions in inquiry-oriented activities. The stats project class is an exemplar of such a reform-oriented classroom.

A norm that is central to the reform-oriented activity structure in the stats project class is that student contributions must, in fact, display reasoning. Let us consider the way the norm is sustained and in this process how it is integral to the activity structure of the stats project class and to children's emerging mathematical goals.

Positions Require Data-Based Justifications

We find consistent evidence of the norm that student contributions required data-based justifications in the stats project class. Kay (the teacher) promoted this norm, and students' mathematical goals took form in relation to it.

In the *Batteries* episode, for example, we find that students were offered turns by Kay that invited their elaboration of positions and justifications for their positions. In taking positions, students created goals of identifying cut points in the display in order to analyze the relative merits of either the *Tough Cell* or *Always Ready* batteries.

Consider the structure of collective activity in which Ceasara asked the teacher to manipulate the range tool on the display depicted in Figure 1 in the Appendix (article by McClain, this issue). Ceasara showed how she came to conclude that *Always Readies* were the best choice. In her talk and references to the display (Figure 1 in the Appendix), Ceasara showed that 7 of the top 10 batteries were *Always Ready* [8–12]. The interchange between Ceasara and Kay was one in which the norm for making justifications explicit was interwoven with Ceasara's goals to quantify, partition, and compare values.

8. Ceasara: OK. You go to the longest lasting battery on the pink. OK, and then, narrow it down to the top 10. Count of 10.
9. Kay: Count of 10; OK, I was wondering if I was supposed to do that. OK.
10. Ceasara: And I was saying see like there's 7 green that last longer.
11. Kay: OK, the greens are the *Always Ready*, so let's make sure we keep up with which set is which, OK?
12. Ceasara: OK, the *Always Ready* is more consistent with the 7 right there, and then like 7 of the *Tough* ones, they're like the...further back...I was just saying 'cuz like all 7, seven out of 3—7 out of 10 of the greens were the longest, and there was...

Here, Ceasara's contribution shows evidence of the position-requires-justification norm, as she worked to support her claim that the "*Always Ready* is more consistent," an activity in which she structures and accomplishes a sequence of goals. At the same time, Kay supported Ceasara's reconstitution of the norm by providing an occasion for Ceasara to make public a particular strategic approach. Subsequently, Jason provided another example of how the norm supports a set of goals. In critiquing Ceasara's position, he made public his case that one should be wary of arbitrary cut points. By sliding the range tool slightly to the left, Jason showed that the difference between the number of *Always Ready* or *Tough Cell* batteries above the threshold line was inconsequential [18].

18. Jason: Ah, see, still, the pink ones, the *Tough Cell*, has more higher ones, like even though it does have more in the end? There's a bunch of close ones in the pink right next, almost in that area. And so then if you put all those in, you'd have 7. (See Figure 1 in the Appendix.)

Finally, Blake took a different perspective. He shifted the criterion for adequacy with his demonstration that if the line was set at 80 in the display, two of the *Always Ready* batteries fell below the threshold, whereas none of the *Tough Cell* batteries fell below this threshold [30].

> 30.Blake: Now, see, there's still green ones behind 80, but all of the *Tough Cell* is above 80. So I'd rather have a consistent battery that I know that'll get me over 80 hr than one that just try to guess.

Again, in Blake's elaborated contribution, we find more evidence of a collective norm for justification and reanalysis, a norm that both sustained the reform-oriented activity structure and supported Blake's own creation of particular mathematical goals.

The *AIDS* episodes contain additional evidence that the norm that contributions should include data-based justifications enables the reform-oriented participation structure, a norm that at the same time has implications for students' emerging goals. Indeed, in each subepisode, students' efforts to evaluate the adequacy of the inscriptions involve justification. In *AIDS* Subepisode 2, for example, Jamie offered a justification in his efforts to identify the relative "majorities of numbers" in the old and new treatment conditions through a partitioning of the data as represented in Inscription 1 in the Appendix [52–57].

> 52. Jamie: I think it's a pretty adequate way of showing information because you can see where the range is starting and ending and you can see where the majority of the numbers are.

In Subepisode 3, we find inscribed in text (Inscription 2) a justification that Kay read [75]: "The new drug was better than the old. The majority of the old ones are behind 550, and the majority of the new drug was in front of 550." In Subepisode 4, Will had difficulties expressing a warrant for his recommendation for the new treatment. However, a careful analysis of his talk reveals his approach [113–118]. He appeared to argue that more than one half of the cases lay below the cut point of 525 for the old treatment; in contrast, this was not the case for the new treatment. In comparing distributions, Will appeared to note that the absolute number of cases was immaterial— the relative properties were what mattered.

In sum, the activity structure of the stats project class is aligned with our current era of educational reform. It is a structure in which the students accomplished much of the critical analysis, while Kay acted as an enabler, working to make visible and support the classroom discussion of valued ideas. It is a structure of activity that has implications for students' emerging mathematical goals, implications that differ from those of traditional classrooms. In making public their approach to analysis, children like Ceasara, Blake, and Jason externalize their thinking, a process that supports revisiting prior goals and means for accomplishing them.

Social Interactions

Social interactions figure prominently in the goals that individuals structure in practices. In the give and take of conversation, one individual's contribution is not only an object of interpretation for another. Another's move may be a call for action, leading to a consideration of grounds for agreement or opposition and a new sequence of goals. In the stats project class, there were many occasions in which interactive moves of participants created occasions for the emergence of mathematical goals.

In the *Batteries* episode, Ceasara's early goal appeared to be to produce an argument supporting the *Always Ready* batteries, a goal that appeared to be sparked by a question from Kay to which we don't have access [1–12].

1. Kay: OK, Ceasara.
2. Ceasara: Could you put the range on?
3. Kay: What do you want me to change it to?
4. Ceasara: Oh, not that, I mean like...[inaudible]
5. Kay: I'm sorry I'm not understanding.
6. Ss: The blue thing.
7. Kay: Ah, count within the range. Sorry. Didn't hear you. Big voice.
8. Ceasara: OK. You go to the longest lasting battery on the pink. OK, and then, narrow it down to the top 10. Count of 10.
9. Kay: Count of 10; OK, I was wondering if I was supposed to do that. OK [places blue range bars in data].
10. Ceasara: And I was saying see like there's 7 green that last longer.
11. Kay: OK, the greens are the *Always Ready*, so let's make sure we keep up with which set is which, OK?
12. Ceasara: OK, the *Always Ready* is more consistent with the 7 right there, and then like 7 of the *Tough* ones, they're like the further back...I was just saying 'cuz like all 7, seven out of 3...7 out of 10 of the greens were the longest, and there was...

Subsequently, Jamie responded to Kay's request to restate Ceasara's argument [14, 15] with reference to the battery display, leading Jamie to create goals involving the reconstruction of Ceasara's rationale that, of the top 10 batteries, 7 were *Always Ready*.

14. Jamie: I understand.

15. Kay: You understand? OK, Jamie, I'm not sure I do. So could you say it for me?

Kay's statement also provoked Jason to consider a new goal: the "what if" consequence of Ceasara's argument [18]—he showed that if the range tool was nudged slightly to the left on the display, the "effect" of battery type disappeared. Jason's finding led Kay to question Ceasara about her initial rationale for choosing the top 10 [19], whereupon Ceasara generated new goals to reconstruct a rationale for her cut point in a multiturn exchange with Kay [19–23].

19. Kay: So you're saying if I open this out a little bit. Well, maybe, Ceasara, you can explain to us why you chose 10. That would be really helpful.
20. Ceasara: All right, there was 10 of the *Always Ready*, and there was 10 of the *Tough*. So that's 20 and half of 20 is 10, so that's how I chose it.
21. Kay: But why would it be helpful for us to know about the top 10? Why did you choose that? Why did you choose 10 instead of 12?
22. Ceasara: Because I was trying to go with the half.
23. Kay: Ah. OK. Blake?

The *AIDS* episode contains many occasions in which mathematical goals were linked to interactional moves of participants. In Kay's introductory move (the first subepisode), she asked the class to consider whether the displays were "an adequate way to represent this data and if we actually understand what folks are doing" [50]. Subsequent uptakes reflected students' efforts to accommodate the question as they created varied data analytic goals. For example, Jaime—the first student to respond to Kay—focused on the range and relative majorities of cases [52].

52. Jamie: I think it's a pretty adequate way of showing information because you can see where the range is starting and ending and you can see where the majority of the numbers are.

In Kay's subsequent moves, she brought forward the contributions of particular students as she revoiced (and restructured) their justifications for particular positions. We can find instances of such revoicing, bringing contributions forward in Kay's pedagogical moves in interactions with Kiri [80, 81].

80. Kiri: Because 550 is in the middle of the whole thing, like the whole, the whole scale. 550 is in the middle. It might not be the middle of the data, but it's the middle of the whole scale.

81. Kay: Oh. So it's like the middle of the range, not necessarily the middle of the... Megan?

We also a find similar revoicing in Kay's uptake on Marissa's contribution [92, 93].

92. Marissa: I would think the second one would be more confusing because it has, since the old program has more numbers than the new program.
93. Kay: Oh. So it looks like that there's more. They had 56 that were above 525, and they only had 37?

Such questioning may well have been a catalyst for subsequent data analytic goals for these (and other) students as Kay brought forward or made efforts to clarify student contributions.

Kay also questioned others' justifications, offering students opportunities to dig deeper, structuring new goals related to creating rationales for their claims. Thus, Kay asked about the criteria for the cutoff points that Caesara used to justify her battery choice [19].

19. Kay: So you're saying if I open this out a little bit. Well, maybe, Ceasara you can explain to us why you chose 10. That would be really helpful.

She put a similar question to Blake, asking him to justify his battery choice [30, 31]:

30. Blake: Now see there's still green ones behind 80, but all of the *Tough Cell* is above 80. So I'd rather have a consistent battery that I know that'll get me over 80 hr than one that just try to guess.
31. Kay: Why? Why were you picking 80?

It was not only the interactive moves by Kay that supported particular kinds of data analytic goals in students' activities. Sometimes children structure goals in relation to one another's contributions, an atmosphere that was supported by Kay's efforts to engage students with one another's arguments. Thus, in the *Batteries* episode, Ceasara's recommendation for the *Always Ready* [12] was critiqued by Jason [18].

18. Jason: Ah, see, still, the pink ones, the *Tough Cell*, has more higher ones, like even though it does have more in the end? There's a bunch of close ones in the pink right next, almost in that area. And so then if you put all those in, you'd have 7.

Ceasara's recommendation for the *Always Ready* batteries [12] also was revoiced by Jamie [16]:

> 16. Jamie: She's understanding, I mean she's saying that out of 10 of the batteries that lasted the longest, 7 of them are green, and that's the most number, so the *Always Ready* batteries are better, because more of those batteries lasted longest.

Also, Jamie questioned Blake's statement about the consistency of the *Always Ready* batteries [34–39].

> 34. Jamie: Um, why wouldn't the *Always Ready* batteries be consistent?
> 35. Blake: Well, because all your *Tough Cell* is above 80, but you still have 2 that are behind 80 in the *Always Ready*.
> 36. Jamie: I know, but that's only 3 out of 10.
> 37. Blake: No, but see, they only did, what 10 batteries? So the 2 or 3 will add up. They'll add up to more bad batteries and all that.
> 38. Kay: Oh, I see; as you get more and more batteries, it's going to get more bad ones if that's representative. OK, is that—Jamie?
> 39. Jamie: So why wouldn't that happen with the *Tough Cell* batteries?

Whether between peers or between teacher and students, in the give and take of interaction, individuals may be pressed in ways that lead them to create new goals in activity, goals that they may not have created on their own.

Artifacts

Artifacts are human constructions. They include material and symbolic forms that permeate our daily lives. In collective practices, some artifacts become valued and used. Such is the case with the Battery display and the AIDS inscriptions. In my remarks that follow, I point to the way in which a few of the many and varied artifacts in the stats project class have implications for the kinds of data analytic goals that children create in activity.

The Tough Cell and Always Ready Display

In the *Batteries* episode, Kay and her students focused on the batteries display artifact. The display, as depicted in Figure 1 in the Appendix, has a number of distinctive properties that are consequential for the character of children's emerging goals. Batteries are represented as 20 distinct cases and cases are grouped by type, *Always Ready* (green, lower distribution) and *Tough Cell* (pink, upper distribution). These properties afford particular kinds of data analytic goals.

Most of the justifications that students offered for their positions were based on counts of batteries, a form of justification afforded by the case-by-case data representation in the display. Recall that Ceasara argued that *Always Ready* batteries were best, because, of the top 10 performing batteries, 7 were *Always Ready* [12]. Jason argued that Caesara's claim ignored the fact that a slight move in the cutoff point would lead to a tie between *Always Ready* and *Tough Cell*, 7 and 7 [18].

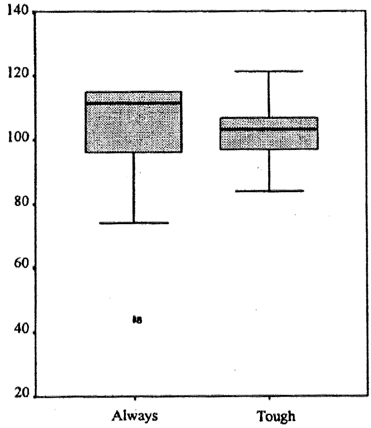
> 18. Jason: Ah, see, still, the pink ones, the *Tough Cell*, has more higher ones, like even though it does have more in the end? There's a bunch of close ones in the pink right next, almost in that area. And so then if you put all those in, you'd have 7.

Also, Blake argued that if one shifted the cut point to 80, the *Tough Cell* should be preferred because 2 *Always Ready* batteries lasted less than 80 hr, whereas none of the *Tough Cell* lasted less than 80 hr [30]. These arguments subsequently were incorporated in the contributions of Jessica [42] and Sequora [48].

> 42. Jessica: I was just going to say that well, even though, 7 of the 10 longest lasting batteries are the *Always Ready* ones, the 2 lowest are also *Always Ready* and if you were using those batteries for something important then you might end up with one of the bad batteries and could [inaudible].

> 48. Sequora: She said that if, even though that the highest 7 were *Always Ready* batteries, the lowest ones were always the *Always Ready* batteries. And if you had something important to do then you could end up with the ones that were the lowest. You know, it'd jeopardize whatever you were gonna do.

Further support for the argument that properties of artifacts have implications for students' goals comes from a thought experiment: Consider what might have occurred had a different display artifact been used than the case-by-case representation of the data display. Like the case-by-case bar chart, a box-and-whisker plot (Figure 1a) would provide access to cases, although the broad distributional characteristics of populations would be highlighted, not specific cases. Thus, in reasoning and argumentation, individuals might well create goals related to the exploration of central tendency in addition to extreme cases. Means and standard deviations (Figure 1b) would "hide" individual cases, perhaps precluding goals that led to arguments of consistency in terms of frequencies; indeed, such a display



a

	Mean	SD	n
Always	100.30	23.65	10
Tough	102.60	10.69	10

b

FIGURE 1 (a) A box-and-whisker plot of the battery data; (b) means, standard deviations, range (minimum, maximum), and number of cases of the battery data.

might well support arguments based on central tendency. Furthermore, some might have pushed for the need for other displays, ones that allowed for exploration of the marked differences between the standard deviations of the two populations. Of course, the problem with such a thought experiment is that the interpretability of the representations as those of battery life would very likely be problematic for children in the stats project class. Box-and-whisker plots and means and standard deviations are compact representations. Both forms of representation are many steps removed from the raw data values. It is no accident that Cobb and colleagues used the case-by-case data presentation in their software as a design feature of their curriculum (see Sfard & McClain, 2002/ this issue).

The T-Cell Counts in the AIDS Displays

A comparative analysis of the AIDS inscriptions presented in Figures 9 through 12 in the Appendix provides additional support that the character of the artifacts have implications for the emergence of particular kinds of mathematical goals.

In the AIDS experiment, the two comparison groups (new treatment and old treatment) contained unequal numbers of patients or an "unbalanced" design (new treatment = 46, old treatment = 186). This is unlike the batteries design, in which groups contained the same number of values (*Always Ready* = 10, *Tough*

Cell = 10). In the various inscriptions, the unbalanced property of the AIDS solutions is revealed or hidden depending on whether case frequencies are cited. For example, no frequencies are cited in Inscriptions 1, 2, and 5 (presented in Figures 9, 10, and 12, respectively, in the Appendix). Instead, AIDS Inscription 1 shows graphically where "most" of the numbers are in each population. AIDS Inscription 2 indicates that the "majority" of the new treatment cases were greater than a cut point and the "majority" of the old treatment cases were less than the same cut point. AIDS Inscription 5 shows a separation of cases into quartiles and the cut points for each quartile. In contrast, the frequencies less than and greater than specified cut points are cited in Inscriptions 3 and 4 (presented in Figures 10 and 11, respectively, in the Appendix). Each reveals that in the old treatment, 56 lie above and 130 lie below the cut point, whereas in the new treatment, 9 cases are less than the same cut point and 37 are greater than that cut point.

What are the consequences for the display of frequencies for children's emerging mathematical goals? For those inscriptions in which frequencies are cited numerically, students may well create goals that involve finding additive differences, framing and structuring a problem solution as the additive difference between the success of the old treatment (56 cases) and the success of the new treatment (37 cases): 56 – 37 = 19 cases. To adjust for differences in sample size (the old treatment contains 186 cases whereas the new treatment contains 46 cases) requires that students create goals that would entail a proportional comparison. They would need to create goals that would allow for a determination of which proportion is greater: 56/186 (old treatment) or 37/46 (new treatment). For those inscriptions that provide summary statements about central tendency that do not include frequencies (e.g., "most," "majority," or graphical depictions), the functional equivalent of a proportional comparison can be produced without necessarily engaging in construction goals that involve proportionality. For example, in Inscriptions 1, 2, and 5 (Figures 9, 10, and 12 in the Appendix), which refer to most or the majority of cases in subsamples, the design may be treated as if it were balanced and the problem solved as it if were an additive one. Marissa [92] voiced this explicitly when she pointed out that Inscription 3 (Figure 10 in the Appendix) was confusing because of the numbers.

92. Marissa: I would think the second one would be more confusing because it has, since the old program has more numbers than the new program.
93. Kay: Oh. So it looks like that there's more. They had 56 that were above 525, and they only had 37?

Indeed, adequate comparisons can be produced for Inscriptions 1, 2, and 5 (Figures 9, 10, and 12, respectively, in the Appendix) without creating goals that involve multiplicative operations, such as structuring a proportional contrast between the data points, 56 and 186 versus 37 and 46.

Prior Understandings

The understandings that children bring to the activity of data analysis are the ground from which children create particular goals. To illustrate, consider four different kinds of understandings that children might use to structure mathematical goals in the batteries activity.

The Batteries Display as Depicting Individual Cases

Some students may not understand principles of hierarchical classification as they pertain to data analysis. Such a student would tend to focus on individual cases, not distinguishing an independent variable and values that such a variable can take on (*Always Ready* and *Tough Cell*). Thus, with such an analytic posture, a student may create goals to consider cases and the length of burning time, not a systematic comparison across subgroups.

The Display as Depicting Groups Composed of Individual Cases

Some students may coordinate superordinate and subordinate classifications in conceptual activities involving data analysis, but those classifications may be limited to populations at hand. In this case, we may find that students consider the display as a depiction of battery lives of two groups of 10 batteries, *Always Ready* and *Tough Cell*, but limit their definitions of their tasks to the batteries at hand.

Some evidence of this understanding is manifested in the remarks of Ceasara [8–12], previously cited. In her activity early in the episode, Ceasara compared the top 10 batteries across the *Tough Cell* and *Always Ready* group and appeared to limit her comparison to the batteries at hand. Jason, also previously cited, noted the arbitrariness of Ceasara's cut point [18]. Students' remarks suggest that their classifications are limited to the particular populations, without a proportional extrapolation to the universe of *Always Ready* and *Tough Cell* batteries.

The Display as Depicting Groups Sampled From Two Populations of Cases

Some students may be competent in proportional thinking with respect to data analysis and may begin to consider problems of inference. Blake appeared to use this type of understanding in his interactions with Jamie when he justified his position related to consistency, arguing that the 2 or 3 bad *Always Ready* batteries below the 80 cut point would "add up" as the sample size increased [34–37]. Blake brought to the task a set of understandings that led him to structure quite different analytic goals than others.

34. Jamie: Um, why wouldn't the *Always Ready* batteries be consistent?
35. Blake: Well, because all your *Tough Cell* is above 80, but you still have 2 that are behind 80 in the *Always Ready*.
36. Jamie: I know, but that's only 3 out of 10.
37. Blake: No, but see, they only did, what, 10 batteries? So the 2 or 3 will add up. They'll add up to more bad batteries and all that.

The Display as Depicting Two Groups Sampled (n = 10) From Two Populations Where n Is a Variable

A student conceivably could have come to the stats project class with some intuitions that might be the basis for an emerging understanding of sampling distributions. In this case, the display might be conceptualized as one of many possible samples where $n = 10$, and that sample would in turn be related to a sampling distribution of many samples of 10 drawn from two universes of *Tough Cell* and *Always Ready* batteries. Such an analytic posture would be grounds for the generation of a quite different set of goals related to experimental reasoning.

Emerging Goals in the *Batteries* and *AIDS* Episodes

So far, my effort has been to show that particular kinds of mathematical goals emerge for children in relation to structures of activity, social interactions, valued artifacts, and children's prior understandings. Although the focus on goals is key to an analysis of children's developing mathematics in practices, in itself it does not provide a framework for analyzing developmental processes that are implicated in the individual's construction of these goals and that flow from the individual's efforts to elaborate and accomplish them.

SECOND STRAND: DYNAMICS OF DEVELOPMENT IN COLLECTIVE PRACTICES

Next, I shift my focus to issues of development. I target the micro-, socio-, and ontogenesis of students' mathematical activities. Each kind of development has its roots in activity as individuals use forms, like the statistical display used in the *Batteries* episodes, to serve varied functions involving data analysis as they structure and accomplish emerging goals. However, although rooted in the same or similar activities, the constructs target differing kinds of developmental processes. Microgenesis is concerned with how particular forms (like the display) and the functions that forms afford are turned into means to accomplish emerging goals in activities. Sociogenesis is concerned with the appropriation and spread of forms in communities, a social process that occurs as individuals appropriate one another's efforts. Ontogenesis is concerned with the shifting relations between the forms used and the functions that they serve in individual activity over an individual's development. I show that a coordinated analysis of these three dimensions of activity is important in explaining students' mathematics as it takes form in the stats project class.

Microgenesis[2]

Cultural artifacts like the battery display and T-cell count inscriptions contain no intrinsic meaning, mathematical or otherwise. Instead of a mathematical object, the battery display might just as well be conceived of as an interesting red and green design; the AIDS Inscription 1 (Figure 10 in the Appendix) could be a depiction of mountain peaks. The assumption that guides the analysis here is that artifacts take on mathematical meaning only in activity, as individuals organize them as means to accomplish particular mathematical goals. This transformation of an artifact into a means to accomplish a goal is a microgenetic process, one in which objects that are not inherently mathematical entities become organized as such in purposeful activity. To analyze microgenetic processes, I focus on the *Batteries* episode and consider three related aspects of students' activities linked to the display form—students' generation of means, goals, and the conceptual operations whereby they create these means and goals. Consider some illustrative instances—one involving Ceasara and Blake.

Ceasara's activity early in the *Batteries* episode can be understood as a microgenetic construction [12], exemplifying the interplay in activity between the emergence of means and goals. In making her case for the *Always Ready* batteries,

[2]The term *microgenesis* has come to be used in two ways in the conceptual and empirical literature on cognitive development. The first meaning dates back to Vygotsky's (1978, 1986) work in the 1930s as well as the later writings of Werner and Kaplan (1962). These authors regarded microgenesis as a developmental process of schematization, either perceptually or conceptually—individuals move from relatively diffuse to more articulated perceptions or conceptualizations over short durations (see also Saxe et al., 1996). More recently, some authors have used microgenesis or microgenetic methods to refer to the study of shifts in children's strategies or cognitive structures over very short periods of time through repeated presentations of similar problems (see, e.g., Miller & Coyle, 1999; Siegler, 1989, 1996, 1997). In this analysis, I use a variation of the first meaning.

Ceasara "specialized" the display form into a means of supporting a recommendation. She did so by isolating the top 10 and noting the differing extensions of the two targeted top-10 subgroups—the *Always Ready* (of which there are 7) versus the *Tough Cell* (of which there are 3).

Inherent in her elaboration of means is an elaboration of goals. The display affords a function of comparing the *Always Ready* and *Tough Cell* batteries with regard to frequencies of burning times, and this function is realized in the comparison goals that she structured—to isolate the top performers and then compare their frequencies.

Ceasara's elaborations of means and goals are grounded in and related through a conceptual activity of partitioning and ordering operations. She appeared to conceptualize a superordinate class of batteries and two partitioned subclasses—the *Always Ready* and the *Tough Cell*—and to also cross-partition these 20 batteries as either longer lasting ($n = 10$) or not longer lasting ($n = 10$). She then ordered the extensions of the longer lasting batteries. These conceptual operations are the basis for the mathematical coherence of her microgenetic constructions of means and goals. Figure 2 contains a schematic of the microgenetic construction.

Blake presented a microgenetic process that supported quite a different outcome, but the broad parameters are similar. Consider his interchange with Kay as he illustrated a partitioning of the data in the display with the range tool [24–33].

24. Blake: Can you put the representative value up there please?
25. Kay: I sure will.

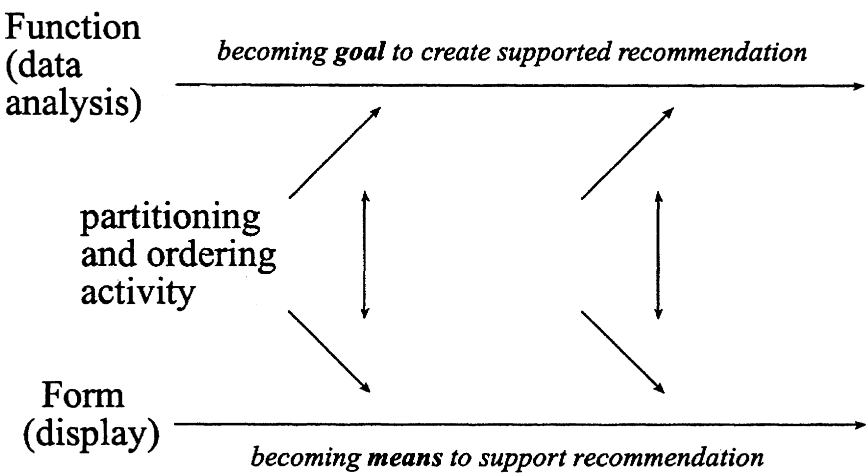


FIGURE 2 Schematic of some Ceasara's microgenetic construction.

26. Blake: Will you put it on 80?
27. Kay: I don't know if I can get it.
28. Blake: That's close.
29. Kay: Is that close enough [drags red value bar into data as shown in Figure 5 in the Appendix]?
30. Blake: Now, see, there's still green ones behind 80, but all of the *Tough Cell* is above 80. So I'd rather have a *consistent* [emphasis added] battery that I know that'll get me over 80 hr than one that just try to guess.
31. Kay: Why? Why were you picking 80?
32. Blake: Well, because most of *Tough Cell* batteries were all over 80.
33. Kay: Ah. OK. So it's like a lower limit for you. OK. Questions for Blake? Yes, Jamie?

In Blake's activity, the display form also became a means to support a recommendation (goal). By creating a cut point at 80, he showed that *Always Ready* batteries were the worst performing batteries. Like Ceasara, he also appealed to consistency, a linguistic form that became a means to support his argument. However, Blake's use of the term *consistency* was different from Ceasara's. Ceasara used *consistency* to refer to the more frequent occurrence of one battery type above a cut point [12]. For Blake, consistency referred to an individual battery, as in a battery that is of "better quality" [40]. Although the word is the same, the function that the word serves and the goals that it is used to support are quite different.

40. Blake: Well, because the way that those 10 batteries show on the chart that they're all over 80 that means that it seems to me that they would have *a better quality* [emphasis added].

Blake also provided evidence of another microgenetic process, one that involves a multiplicative extension of the partitioning operation. He treated the display form as a representation of cases drawn from a larger universe of batteries. Blake used the relative frequencies in the *Always Ready* and *Tough Cell* displays as a means to argue for the value of the *Tough Cell* by extending the products of his partitioning as a proportional relation to a larger universe of *Always Ready* and *Tough Cell* samples [35–37].

35. Blake: Well because all your *Tough Cell* is above 80, but you still have 2 that are behind 80 in the *Always Ready*.
36. Jamie: I know, but that's only 3 out of 10.
37. Blake: No, but see, they only did, what, 10 batteries? So the 2 or 3 will add up. They'll add up to more bad batteries and all that.

Treating the display in proportional terms, Blake turned the display into a means for extrapolating to additional samples of batteries, further supporting his data analytic goal.

Particularly noteworthy in children's activities is the property of emergence in microgenetic processes. The data display can only become a means to support a particular recommendation if that recommendation is known at the outset. But in the case of experimental data analysis, the recommendation is not known at the outset. At the same time, a recommendation needs a data-based justification to support it. However, a means emerges in activity and is not given at the outset. The *Batteries* and *AIDS* episodes provide repeated evidence of a bootstrapping in processes of microgenesis, a bootstrapping between the form and "received" function of artifacts as they were organized and reorganized into mathematical means and goals though conceptual activity.

Sociogenesis

Sociogenesis involves the process of emergence and travel of new and valued forms in the history of a practice. Like microgenetic processes, sociogenetic processes are pervasive in activity. The sociogenesis of mathematical activity occurs as individuals appropriate means that others have generated for accomplishing goals, folding these means into their own microgenetic constructions in collective practices.

In the stats project class, children produced uptakes on one another's contributions that sometimes entailed using the same or similar word forms. Good examples include the use of expressions like "majority" and "most of the numbers" in the AIDS clips. Consider, for example, Jamie's contribution [52]:

> 52. Jamie: I think it's a pretty adequate way of showing information because you can see where the range is starting and ending and you can see where the majority of the numbers are.
> 59. Jamie: Where most of the numbers were.

Sheena [64] also used of the term *majority*:

64. Sheena: Yeah, like right in there, that's where the majority of it is.

Derrick [60] referred to a similar notion:

60. Derrick: Where most of the numbers are...

Finally, the written artifact in Inscription 2 (Figure 10 in the Appendix) also referred to a majority.

75. Kay: The new drug was better than the old. The majority of the old ones are behind 550, and the majority of the new drug was in front of 550.

Another example of a word that was appropriated or used in different ways by children is the term *consistency*. This term was used by Ceasara [12], Blake [30], and Jamie [34]. Let's consider *consistency* in a little more detail as the word initially was used by Ceasara and later appropriated by Blake to serve a different function.

Ceasara introduced the term in her claim that she would prefer the more consistent battery, where consistency is a group characteristic [12]. She preferred *Always Ready* because they were consistently high (7 of the top 10). Blake brought the word forward late in the clip, when he indicated that he also preferred a consistent battery. Here, in contrast, *consistency* was used to indicate a characteristic of an individual battery—one of "good quality" that didn't have a burn time below 80 [30]. Nonetheless, there apparently was enough overlap in the ways the words were used to preserve the coherence of the classroom discourse, even though there was uptake by Jamie showing that she was confused by Blake's intended meaning [33–34].

33. Kay: Ah. OK. So it's like a lower limit for you. OK. Questions for Blake? Yes, Jamie?
34. Jamie: Um, why wouldn't the *Always Ready* batteries be *consistent*?

Were *consistency* to become a marked term in the stats project class with a technical meaning for the group, we might well find continued discourse to "fix" a meaning. If this became identified as a value in the community in discussing the characteristics of data inscriptions, we might well find that *consistency* becomes an artifact that has important implications for students' data analytic goals.

Another sociogenetic process in the classroom involves students' use of cut points to justify their positions. Initially, Ceasara argued for a cut point that should be determined by the top 10 [12]. Later, Jason argued that the cut point distorted the minor differences across distributions, pointing out that moving the cut point only slightly led to the same number of each battery type above the cut point [18]. Finally, Blake argued for the use of a cut point at the lower end of the distribution, pointing out that if the cut point were set at 80, two of the *Always Ready* but none of the *Tough Cell* batteries would fall below it [30].

The sociogenesis of mathematical activity in the stats project class was supported not only in peer uptakes on one another's arguments, but also by Kay herself as she revoiced student contributions. For example, she restated Jason's remarks about the arbitrariness of cut points, redirecting the remark back to Ceasara [19].

19. Kay: So you're saying if I open this out a little bit. Well, maybe, Ceasara, you can explain to us why you chose 10. That would be really helpful.

She also revoiced Blake's argument for setting a cut point at 80 [38].

38. Kay: Oh, I see; as you get more and more batteries, it's going to get more, more bad ones if that's representative.

On these occasions, children's arguments lived beyond their own constructions, becoming means for Kay to accomplish her own pedagogical goals. In her transformation of the voices of others into means to accomplish her own goals, Kay took on a powerful role in shaping the classroom discussion.

Ontogenesis

Children, over time, develop new functions for forms in their activities. They also may appropriate new forms, using them for earlier understood functions. The interplay between the forms and functions over the course of children's developments is a process of ontogenetic change and key to understanding children's developing mathematics in collective practices.

Of course, the most useful source of data for ontogenetic analyses is to follow individuals as they grow older, sampling changes in the ways that individuals structure and accomplish recurring problems. This is not possible given the available corpus of episodes in the stats project class. It is possible to remark on some prototypical approaches that students used in the episodes and whether these approaches might be ordered over individual children's developments. Consider two general functions described earlier for which the battery display could be used that may well constitute a very general developmental trajectory.

In the stats project class, many students appeared to use the display to accomplish a descriptive function. Using the batteries display form, students were counting the number of batteries that were above a cut point or the number of batteries that were below a cut point without clear regard for how such comparisons would generalize to a universe of batteries. The anomaly was Blake, who, to defend his position for the *Tough Cell* batteries, created a multiplicative relation as he considered the extrapolation of the current sample of *Tough Cell* and *Always Ready* batteries to additional samples [35–37].

35. Blake: Well, because all your *Tough Cell* is above 80, but you still have 2 that are behind 80 in the *Always Ready*.
36. Jamie: I know, but that's only 3 out of 10.
37. Blake: No, but see, they only did, what, 10 batteries? So the 2 or 3 will add up. They'll add up to more bad batteries and all that.

In this effort, Blake created a new function for the display form, a function that differed from other children's functions. Blake used the form to serve as a basis for extrapolation of a proportional relation as he accomplished the goal of supporting his preference for the *Tough Cell* batteries. What marks Blake's approach is the multiplicative extrapolation of the partitioning to unobserved cases.

There is some evidence of the use of proportional reasoning in the later occurring *AIDS* episodes by other members of the class. Indeed, of the students who contributed to the AIDS discussion, many voiced analyses that appeared to appreciate the need to structure multiplicative approaches to the analyses of the display. They pointed out that although the old treatment was associated with cases with greater T-cell counts, the new treatment had the greater proportion of T-cell counts. Paul and Kay even pushed the children to adopt absolute (additive) as contrasted to multiplicative approaches, but they had no takers [99, 113–114].

> 99. Paul: I've got a question for everybody. Couldn't you just argue, hey, this shows really convincingly that the old treatment was better, right? Because there were 56 of them, 56 scores above 525, 56 people with T-cell counts above 525, and here there's only 37 above, so the old one just had to be better, there's more people. I mean, there's 19 more people in there, so that's the better one, surely.
>
> 113. Kay: OK, who can help me out with that, who can say that a different way so that I might could understand that? Will, can you say it a different way?
>
> 114. Will: Well, in that situation it wouldn't matter how many people were in there because see like...

That Kay and Paul had no takers for their "absolute value" arguments suggests that many of the children were persuaded by the relative arguments advanced by the children who contributed to the classroom discussion.

It is noteworthy that properties of the designed artifacts in the stats project class lessons are sequenced in such a way that parallel the ontogenetic shifts in children's activities. In the *Batteries* episode, the numbers of cases in the populations were small and the design was balanced—the number of *Tough Cell* and *Always Ready* batteries both equaled 10 (as depicted in the software display). Furthermore, in the *Batteries* episode, each case was represented as a distinct bar, a bar that was a "motivated" representation in the sense that the greater the battery life, the greater the length of a batteries bar. Thus, the forms of representation well support functions of considering individual cases in relation to others or computing the two groups of 10 batteries. In contrast, in the *AIDS* episode inscriptions, the numbers of cases was relatively large and the design was unbalanced ($n = 46$ and $n = 186$ for the new and old treatments, respectively). In the *AIDS* episodes, students were asked to create their own abbreviated representation of the software output, and the adequacy of these representations was the focus of discussion so that someone else could make a judgment about treatment—a problem of inference. Furthermore, data were not represented as single instances but rather the representation of cases was accomplished in abbreviated ways. Thus, these data and their representations "pulled" for a multiplicative analysis.

Putting It Together

Figure 3 contains an overview schematic of the cultural developmental framework that I've sketched in my remarks. The figure portrays two children. As these two children engage with data analytic problems that take form in the stats project class, they structure resources (such as statistical display forms, linguistic forms of statistical terms) into means of solution to accomplish emerging goals in their own microgenetic constructions. These constructions are depicted as the ovals in the figure. Sometimes, depicted as the overlap of the ovals, the children may appropriate the fruits of the others's productions—the means that the other creates in his or her microgenetic construction. As children come to appropriate one another's efforts, processes of microgenesis may build off of one another. Such "travel" of means to serve sometimes similar and sometimes different functions in reciprocal appropriations is the root of the sociogenesis of knowledge in collective practices. As suggested by Figure 3, uptake by additional others may lead to an eventual "institutionalization" of such means as the "received" or "reified" convention. The diagram also situates ontogenetic developments—shifts in the organization of individuals' continued efforts to accomplish similar problems from one time to another—in relation to micro- and sociogenetic processes. Indeed, microgenetic constructions may take on new qualities as individuals use familiar forms to serve newly emerging functions or use newly introduced forms for functions that either are familiar or will only be realized in later activity.

CONCLUDING REMARKS

A caveat is in order. In this article I supported the elaboration of key constructs with evidence from observations drawn from selected videotaped classroom episodes—a method that is a departure from the multiple methods I regard as important in research on cognition in collective practices. Indeed, in this exercise, I made claims about children's goals, the means that they used to accomplish goals, and developmental processes, citing actions and utterances in the videotaped records as

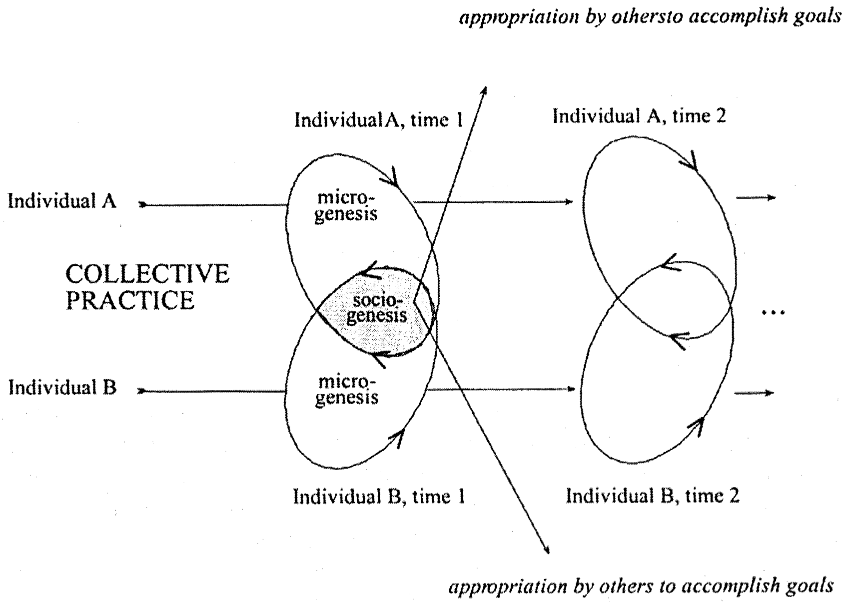


FIGURE 3 General portrayal of the interplay between micro-, socio-, and ontogenetic processes.

my sole support. Were I to design a study on collective practices like the stats project class with a concern for the constructs that I have described, I might well have gathered additional data sources to provide convergent evidence. This is the approach that I have taken in my research on collective practices, whether in early studies of young children's activities involving numbers in middle- and working-class communities in the United States (Saxe et al., 1987), the mathematics of Brazilian candy sellers (Saxe, 1991), or recent studies on mathematics and historical change in remote Papua New Guinea communities (Saxe, Esmonde, & McIntosh, 2002). The framework that I have sketched in this article has its roots in and is used to structure such multimethod analyses.

In an analysis of the means and goals that children were generating in the stats project class, one might have found helpful supplemental information gleaned from "debriefings" of children after targeted lessons. To support analyses of micro- and ontogenetic changes in children's representational activities, I would have found useful interviews with students on problems closely related to classroom life. For analyses of sociogenesis, debriefings with the silent children would have been useful to understand the sense that they were making of their peers' conjectures and their teacher's moves. In addition, surveying the children on the sources of particular strategies would produce information on the spread of forms and the functions that they were used to serve in the stats project class community.

Any video record of activity in a mathematics classroom, regardless of how rich, is open to multiple interpretations. This issue illustrates the importance and utility of analysts' epistemological assumptions in framing treatments of teaching and learning based on such records. It also illustrates the need for multiple empirical techniques to advance, support, and constrain important claims about developmental processes related to ongoing classroom activity.

ACKNOWLEDGMENTS

The preparation of this article was made possible in part by a grant from the Spencer Foundation. The statements made and the views expressed are solely the responsibility of the author.

Appreciation is extended to Maryl Gearhart, Britte Cheng, Rachel Coben, Jenny Garcia de Osuna, Cliff McIntosh, Behnaz Shahidi, Edd Taylor, Anna Sfard, and Kay McClain for editorial comments on earlier versions of the text.

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